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Electric and magnetic fields of two infinitely long parallel cylindrical conductors carrying a DC current*

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Abstract. This paper calculates the electric and magnetic fields and the Poynting vector around two infinitely long parallel cylindrical conductors, carrying a DC current. Also the charges on the surface of the wire are calculated, and their distribution is visualized. The wire is assumed to be perfectly electrically conducting. Furthermore, the Hall effect is ignored. In the literature [S.J. Orfanidis, Electromagnetic waves and antennas, 2008], the problem of determining the electric field is usually tackled using an equivalent model consisting of two line charge densities, producing the same electric field. In this work, the Laplace equation is rigorously solved. The authors found no work explaining the solution of the Laplace equation with boundary conditions for this problem and hence thought it was useful to dedicate a paper to this topic. The method of separation of variables is employed and a bipolar coordinate system is used. After solving the appropriate Sturm-Liouville problems, the scalar potential is obtained. Taking the gradient yields the electric field.

1 Problem description

Two cylindrical wires, each conducting a DC current I, are parallel to the z-axis (Fig. 1). The first wire is located at $x = -d$ and conducts the current in the positive z-direction. The other wire is located at $x = d$ and conducts the current in the negative z-direction. The left wire has a potential V_1 and the right wire $-V_1$. We wish to determine the scalar potential, the electric and the magnetic field in the region outside the conductors. The cross-section of the right wire is bounded by a circle, called C_1. The wires are assumed to be perfectly electrically conducting, that is they have zero resistivity.

The electric potential ϕ is given by the Laplace equation:

$$\nabla^2 \phi = 0. \tag{1}$$

We will solve this equation using an appropriate coordinate system: bipolar coordinates.

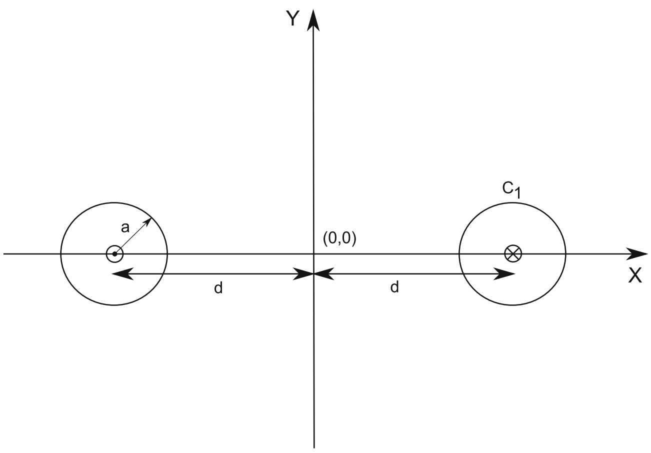

Fig. 1. Geometry of the problem.

2 Bipolar coordinates

The bipolar coordinate system [4] is defined as ($\tau \in (-\infty, \infty)$ and $\sigma \in [0, 2\pi]$ and $\neg(\sigma = \tau = 0)$):

$$x = \alpha \frac{\sinh \tau}{\cosh \tau - \cos \sigma}, \tag{2}$$

$$y = \alpha \frac{\sin \sigma}{\cos \tau - \cos \sigma}. \tag{3}$$

The curves of constant σ form non-concentric circles centered on the y-axis:

$$x^2 + (y - \alpha \cot \sigma)^2 = \frac{\alpha^2}{\sin^2 \sigma}. \tag{4}$$

The radius decreases as σ increases in the interval $[0, \pi/2]$ and increases again with increasing σ in the interval $[\pi/2, \pi]$. The curves of constant τ form non-concentric circles centered on the x-axis:

$$(x - \alpha \coth \tau)^2 + y^2 = \frac{\alpha^2}{\sinh^2 \tau}. \tag{5}$$

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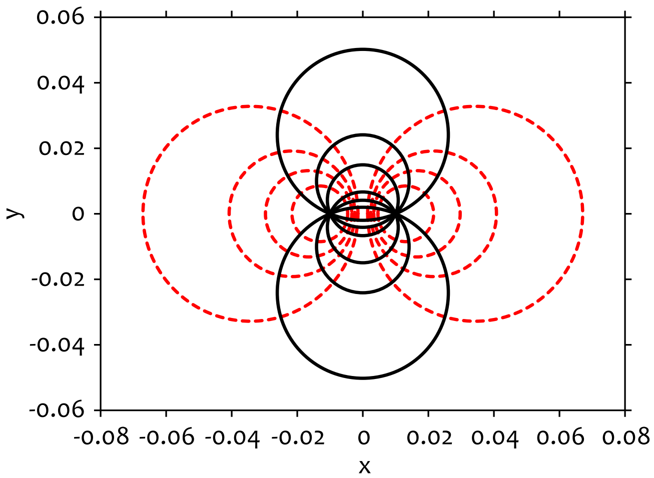

Fig. 2. Bipolar coordinate system: curves of constant τ (dashed) and constant σ (solid).

The radius decreases as τ increases from $-\infty$ to 0 and increases as τ increases from 0 to ∞. For $\tau < 0$ the circles lie in the $x < 0$ half-plane, for $\tau > 0$ the circles lie in the $x > 0$ half-plane.

The circles of constant τ respectively constant σ are depicted in Figure 2.

For the problem of determining the electric scalar potential in the region outside the two wires, also the boundary conditions are rewritten in terms of bipolar coordinates.

3 Boundary conditions

The electric scalar potential on the surface of the wires is known from the problem description: $\phi = V_1$ on the circumference of the circle C_1, with center $(0, d)$ and radius a. C_1 coincides with a circle of constant τ (cf. (5), [4]), if

$$\begin{cases} d = \alpha \coth \tau, \\ a = \dfrac{\alpha}{\sinh \tau}. \end{cases} \tag{6}$$

We thus have:

$$d = \alpha \coth \tau = \underbrace{\frac{\alpha}{\sinh \tau}}_{=a} \cosh \tau = a \cosh \tau.$$

Let us call the τ for which this circle C_1 is defined, τ_c. We have $\cosh \tau_c = d/a$, and C_1 is thus defined by the single τ-coordinate:

$$\tau_c = \cosh^{-1} \frac{d}{a}. \tag{7}$$

We can also calculate α from the second equation of (6):

$$a^2 = \frac{\alpha^2}{\sinh^2 \tau_c} = \frac{\alpha^2}{\cosh^2 \tau_c - 1} = \frac{\alpha^2}{\left(\frac{d^2 - a^2}{a^2}\right)}.$$

Thus:

$$\alpha = \sqrt{d^2 - a^2}. \tag{8}$$

Similarly, the surface of the second conductor corresponds to points in the (τ, σ)-plane with $\tau = -\tau_c$. Here, $\phi = -V_1$.

4 Formulation of the problem in bipolar coordinates

The scalar potential ϕ has to obey the Laplace partial differential equation in bipolar coordinates [2]:

$$\nabla^2 \phi = \frac{1}{\alpha^2} (\cosh \tau - \cos \sigma)^2 \left(\frac{\partial^2 \phi}{\partial \sigma^2} + \frac{\partial^2 \phi}{\partial \tau^2} \right) = 0. \tag{9}$$

Therefore, as long as $\neg(\sigma = \tau = 0)$,

$$\frac{\partial^2 \phi}{\partial \sigma^2} + \frac{\partial^2 \phi}{\partial \tau^2} = 0, \tag{10}$$

with boundary conditions:

$$\phi(\sigma, \tau = \tau_c) = V_1, \qquad \sigma \in [0, 2\pi], \tag{11}$$

$$\phi(\sigma, \tau = -\tau_c) = -V_1, \qquad \sigma \in [0, 2\pi], \tag{12}$$

where ϕ is a periodic function in σ with a period equal to 2π, and τ_c as defined in (7).

In order for a particular boundary value problem to be solvable using the method of separation of variables, certain conditions have to be met. These conditions involve (a) the differential equation itself as well as (b) the shape of the boundary and (c) the form of the boundary conditions (cf. [3], p. 68). It can be easily verified that (10) is separable (condition (a)). Since the boundary conditions are imposed on coordinate lines and are constants, conditions (b) and (c) are fulfilled.

5 Solution of the Laplace equation with boundary conditions

The method of separation of variables is applied. This means that the solution for the potential ϕ is written as the product of single-variable functions:

$$\phi(\sigma, \tau) = S(\sigma) T(\tau). \tag{13}$$

When we fill this equation into the Laplace equation (9), we must solve a system of two differential equations:

$$S''(\sigma) - k S(\sigma) = 0, \tag{14}$$

$$T''(\tau) + k T(\tau) = 0. \tag{15}$$

This has different solutions depending on the sign of k.

5.1 Case k = 0

If $k = 0$ then (14) and (15) become

$$S''(\sigma) = 0,$$
$$T''(\tau) = 0.$$

The solutions for which are:

$$T(\tau) = A_0 + A_1 \tau, \tag{16}$$

$$S(\sigma) = B_0 + B_1 \sigma. \tag{17}$$

But because $S(\sigma)$ is periodic in σ, we must demand that $B_1 = 0$. We thus have:

$$T(\tau) = A_0 + A_1 \tau, \tag{18}$$

$$S(\sigma) = B_0. \tag{19}$$

5.2 Case k > 0

If $k > 0$, then the general particular solution of (14) is:

$$S(\sigma) = C_1 \cos(\sqrt{k}\sigma) + C_2 \sin(\sqrt{k}\sigma).$$

However, because $S(\sigma)$ is periodic in σ with period 2π, $\sqrt{k} = n$ where n is a whole number, greater than 0. The solution for $S(\sigma)$ is thus:

$$S(\sigma) = D_n \cos(n\sigma) + E_n \sin(n\sigma). \tag{20}$$

The solution for $T(\tau)$ is:

$$T(\tau) = F_n \cosh(n\tau) + G_n \sinh(n\tau). \tag{21}$$

5.3 Case k < 0

If $k < 0$ then the general solution of (14) is:

$$S(\sigma) = C_3 \cosh(\sqrt{k}\sigma) + C_4 \sinh(\sqrt{k}\sigma).$$

However, because $S(\sigma)$ is a periodic function in σ, this solution is not feasible. We can only allow non-negative k's.

5.4 Solution of ϕ

The scalar potential $\phi(\sigma, \tau)$ is a linear combination of the solutions for every k-value and has the form:

$$\phi(\sigma,\tau) = (A_0 + A_1\tau)B_0 + \sum_{n=1}^{\infty} [(D_n \cos(n\sigma) + E_n \sin(n\sigma))(F_n \cosh(n\tau) + G_n \sinh(n\tau))]\,. \tag{22}$$

We now impose the remaining boundary conditions (11) and (12):

5.4.1 Use of boundary condition (11)

Imposing boundary condition (11) on (22) leads to:

$$\phi(\sigma, \tau = \tau_c) = (A_0 + A_1\tau_c)B_0 + \sum_{n=1}^{\infty} [D_n(F_n \cosh(n\tau_c) + G_n \sinh(n\tau_c)) \cos(n\sigma) + E_n(F_n \cosh(n\tau_c) + G_n \sinh(n\tau_c)) \sin(n\sigma))] = V_1. \tag{23}$$

This is a Fourier series of the constant function V_1. Equating the corresponding coefficients results in:

$$(A_0 + A_1\tau_c)B_0 = V_1, \tag{24}$$

$$D_n(F_n \cosh(n\tau_c) + G_n \sinh(n\tau_c)) = 0 \quad \forall n \geq 1, \tag{25}$$

$$E_n(F_n \cosh(n\tau_c) + G_n \sinh(n\tau_c)) = 0 \quad \forall n \geq 1. \tag{26}$$

5.4.2 Use of boundary condition (12)

Imposing boundary condition (12) on (22) leads to:

$$\phi(\sigma, \tau = \tau_c) = (A_0 - A_1\tau_c)B_0 + \sum_{n=1}^{\infty} [D_n(F_n \cosh(n\tau_c) - G_n \sinh(n\tau_c)) \cos(n\sigma) + E_n(F_n \cosh(n\tau_c) - G_n \sinh(n\tau_c)) \sin(n\sigma))] = -V_1. \tag{27}$$

This is a Fourier series of the constant function $-V_1$. Equating the corresponding coefficients results in:

$$(A_0 - A_1\tau_c)B_0 = -V_1, \tag{28}$$

$$D_n(F_n \cosh(n\tau_c) - G_n \sinh(n\tau_c)) = 0 \quad \forall n \geq 1, \tag{29}$$

$$E_n(F_n \cosh(n\tau_c) - G_n \sinh(n\tau_c)) = 0 \quad \forall n \geq 1. \tag{30}$$

5.4.3 Solving for the coefficients of linear combination (22)

Adding and subtracting equations (24) and (28) gives:

$$A_0B_0 = 0, \tag{31}$$

$$A_1B_0\tau_c = V_1. \tag{32}$$

Similarly (25) and (29) result in:

$$D_nF_n = 0 \quad \forall n \geq 1, \tag{33}$$

$$D_nG_n = 0 \quad \forall n \geq 1. \tag{34}$$

And finally from (26) and (30) it follows that:

$$E_nF_n = 0 \quad \forall n \geq 1, \tag{35}$$

$$E_nG_n = 0 \quad \forall n \geq 1. \tag{36}$$

Using these expressions, ϕ can be written as:

$$\phi(\tau) = \frac{V_1}{\tau_c}\tau. \tag{37}$$

From the definition of the bipolar coordinate system, equation (2), it can be seen that the y-axis, for which $x = 0$, corresponds to points in the (τ, σ)-plane with $\tau = 0$. Hence, it is interesting to note that, as could be expected from the symmetry of the problem, $\phi(x = 0) = \phi(\tau = 0) = 0$. However, the presented solution did not assume any prior knowledge about electrostatic field solutions.

6 Expressing the scalar potential in Cartesian coordinates

Making use of (5), and the identity $\cosh^2\tau - \sinh^2\tau = 1$ we get:

$$
\begin{aligned}
(x-\alpha\coth\tau)^2 + y^2 &= \frac{\alpha^2}{\sinh^2\tau} \\
&= \frac{\alpha^2}{\sinh^2\tau}(\cosh^2\tau - \sinh^2\tau) \\
&= \alpha^2\coth^2\tau - \alpha^2 \\
&\Downarrow \\
x^2 - 2\alpha x\coth\tau + \alpha^2\coth^2\tau + y^2 &+ \alpha^2 - \alpha^2\coth^2\tau = 0 \\
\Longrightarrow \coth\tau &= \frac{x^2+y^2+\alpha^2}{2\alpha x}.
\end{aligned}
$$

And thus:

$$\tau = \tanh^{-1}\left(\frac{2\alpha x}{x^2+y^2+\alpha^2}\right). \tag{38}$$

The complete solution for the scalar potential in Cartesian coordinates is thus:

$$
\begin{aligned}
\phi(x,y) &= \frac{V_1}{\cosh^{-1}\frac{d}{a}}\tanh^{-1}\left(\frac{2x\sqrt{d^2-a^2}}{x^2+y^2+d^2-a^2}\right) \\
&= \frac{V_1}{\cosh^{-1}\frac{d}{a}}\ln\sqrt{\frac{x^2+y^2+\alpha^2+2\alpha x}{x^2+y^2+\alpha^2-2\alpha x}}.
\end{aligned}
$$

$$\phi(x,y) = \frac{V_1}{\ln\left(\frac{d}{a}+\sqrt{\frac{d^2}{a^2}-1}\right)}\ln\sqrt{\frac{(x+\sqrt{d^2-a^2})^2+y^2}{(x-\sqrt{d^2-a^2})^2+y^2}}. \tag{39}$$

7 Electric field

Because the wires are assumed to be infinitely long and to have zero resistivity, the electric field is independent of the z-coordinate. The electric field outside the two conducting wires is thus:

$$\boldsymbol{E}(x,y) = -\boldsymbol{\nabla}\phi \tag{40}$$

$$\boldsymbol{E} = \frac{2(x^2-d^2+a^2-y^2)\sqrt{d^2-a^2}V_1}{N}\times\boldsymbol{e}_x + \frac{4V_1xy\sqrt{d^2-a^2}}{N}\times\boldsymbol{e}_y, \tag{41}$$

where

$$N = ((x+\sqrt{d^2-a^2})^2+y^2)\ln\left(\frac{d}{a}+\sqrt{\frac{d^2}{a^2}-1}\right) \times((x-\sqrt{d^2-a^2})^2+y^2). \tag{42}$$

8 Magnetic field

A DC current I flows in the positive z-direction in the wire at $x=-d$, and a DC current of I flows in the negative z-direction in the wire at $x=d$.

Call H_1 the magnetic field outside the two conductors, produced by the conductor at $x=-d$ and H_2 the magnetic field outside the two conductors, produced by the conductor at $x=d$. We have:

$$H_1 = \frac{I}{2\pi\sqrt{(x+d)^2+y^2}}, \tag{43}$$

$$H_2 = \frac{I}{2\pi\sqrt{(-x+d)^2+y^2}}. \tag{44}$$

8.1 Magnetic field in the region between the two wires: −d ≤ x ≤ d

With θ the acute angle between the x-axis and the line from $(-d,0)$ to (x,y) and γ the acute angle between the x-axis and the line from $(d,0)$ to (x,y), we have:

$$\theta = \arctan\frac{y}{x+d}, \tag{45}$$

$$\gamma = \arctan\frac{y}{-x+d}, \tag{46}$$

and

$$\boldsymbol{H} = -H_1\sin\theta\boldsymbol{e}_x + H_1\cos\theta\boldsymbol{e}_y + H_2\sin\gamma\boldsymbol{e}_x + H_2\cos\gamma\boldsymbol{e}_y.$$

Making use of the identities $\sin(\arctan(x)) = x/\sqrt{1+x^2}$ and $\cos(\arctan(x)) = 1/\sqrt{1+x^2}$, we obtain for the magnetic field outside the two wires, neglecting the small Hall effect due to which the current density is not exactly constant in each wire:

$$\boldsymbol{H}(x,y) = \frac{Iy}{2\pi}\left(\frac{-1}{(x+d)^2+y^2}+\frac{1}{(x-d)^2+y^2}\right)\boldsymbol{e}_x + \frac{I}{2\pi}\left(\frac{(x+d)}{(x+d)^2+y^2}+\frac{-(x-d)}{(x-d)^2+y^2}\right)\boldsymbol{e}_y. \tag{47}$$

We can see that inside this region, the dot product of the electric and the magnetic field is equal to:

$$\boldsymbol{E}\cdot\boldsymbol{H} = \frac{4Ida^2V_1xy\sqrt{(}d^2-a^2)/\left(\pi\ln\left(\frac{d}{a}+\sqrt{\frac{d^2}{a^2}-1}\right)\right)}{N_2}, \tag{48}$$

where

$$N_2 = ((x+d)^2+y^2)((x-d)^2+y^2) \times((x+\sqrt{d^2-a^2})^2+y^2)((x-\sqrt{d^2-a^2})^2+y^2). \tag{49}$$

This means that the electric and magnetic fields are not orthogonal in this region, except on the x- and y-axes.

8.2 Magnetic field in the region for which x $\leq -$d

With θ the acute angle between the x-axis and the line from $(-d,0)$ to (x,y) and γ the acute angle between the x-axis and the line from $(d,0)$ to (x,y), we have:

$$\theta = \arctan\frac{-y}{x+d}, \tag{50}$$

$$\gamma = \arctan\frac{y}{-x+d}, \tag{51}$$

and

$$\boldsymbol{H} = -H_1\sin\theta\boldsymbol{e}_x - H_1\cos\theta\boldsymbol{e}_y + H_2\sin\gamma\boldsymbol{e}_x + H_2\cos\gamma\boldsymbol{e}_y. \tag{52}$$

Making use of the identities $\sin(\arctan(x)) = x/\sqrt{1+x^2}$ and $\cos(\arctan(x)) = 1/\sqrt{1+x^2}$, we can again express the magnetic field in Cartesian coordinates. However, the expression is so long that for numerically evaluating it, we recommend that equation (52) is used.

The dot product of the electric and the magnetic field can again be calculated. It is never zero in this region except on the x-axis. This means that the electric and magnetic fields are not orthogonal in this region, except on the x-axis.

8.3 Magnetic field in the region for which x $\geq$ d

With θ the angle between the x-axis starting at $(-d,0)$ and going to plus infinity and the line from $(-d,0)$ to (x,y) and γ the angle between the x-axis starting at $(d,0)$ and going to plus infinity and the line from $(d,0)$ to (x,y), we have:

$$\theta = \arctan\frac{y}{x+d}, \tag{53}$$

$$\gamma = \arctan\frac{y}{x-d}, \tag{54}$$

and

$$\boldsymbol{H} = -H_1\sin\theta\boldsymbol{e}_x + H_1\cos\theta\boldsymbol{e}_y + H_2\sin\gamma\boldsymbol{e}_x - H_2\cos\gamma\boldsymbol{e}_y. \tag{55}$$

Making use of the identities $\sin(\arctan(x)) = x/\sqrt{1+x^2}$ and $\cos(\arctan(x)) = 1/\sqrt{1+x^2}$, we can again express the magnetic field in Cartesian coordinates. However, the expression is so long that for numerically evaluating it, we recommend that equation (55) is used.

The dot product of the electric and the magnetic field can again be calculated. It is never zero in this region except on the x-axis. This means that the electric and magnetic fields are not orthogonal in this region, except on the x-axis.

9 Surface charges on the wires

The surface density of the free charges on the surface of the right wire, centered around $x = d$, is σ_R, and is equal

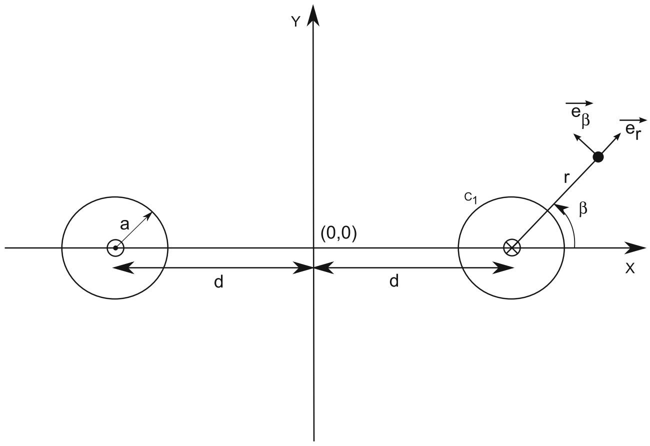

Fig. 3. Polar coordinate system, used for determining the surface charge density.

to ϵ_0 times the normal component of the electric field, perpendicular to the circle C_1. Here, ϵ_0 is the permittivity of vacuum. The surface density of the charges on the surface of the left wire is then, by symmetry, the mirrored image and the negative of σ_R. Let us determine σ_R. We choose a polar coordinate system (r,β) (Fig. 3) where r is the distance from $(x,y) = (d,0)$ to the observed point, and β is the angle between the x-axis, pointing from $(x,y) = (d,0)$ to infinity and the line from $(x,y) = (d,0)$ to the observed point. We have, for points on circle C_1, with $\beta \in [0,2\pi]$:

$$x = d + a\cos\beta, \tag{56}$$

$$y = a\sin\beta, \tag{57}$$

$$\boldsymbol{e}_x = \cos\beta\boldsymbol{e}_r - \sin\beta\boldsymbol{e}_\beta, \tag{58}$$

$$\boldsymbol{e}_y = \sin\beta\boldsymbol{e}_r + \cos\beta\boldsymbol{e}_\beta. \tag{59}$$

Making these substitutions, we find a very simple form for the radial component of the electric field:

$$E_r(r=a,\beta) = \frac{V_1\sqrt{d^2-a^2}}{a(d+a\cos\beta)\ln\left(\frac{d}{a}+\sqrt{\frac{d^2}{a^2}-1}\right)}.$$

Hence, the surface charge density [C/m^2] is:

$$\sigma_R = \epsilon_0 E_r(r=a,\beta) = \frac{\epsilon_0 V_1\sqrt{d^2-a^2}}{a(d+a\cos\beta)\ln\left(\frac{d}{a}+\sqrt{\frac{d^2}{a^2}-1}\right)}. \tag{60}$$

This corresponds with the expression derived with the equivalent line charge model, in [5].

10 Visualisation of solutions

With $I = 1$ A, $V_1 = 0.5$ V, $d = 0.01$ m and $a = 0.5$ mm, the following figures depict, with some contours, the Poynting field (Fig. 4), the electric field (Fig. 5), the magnetic field (Fig. 6) and the scalar potential

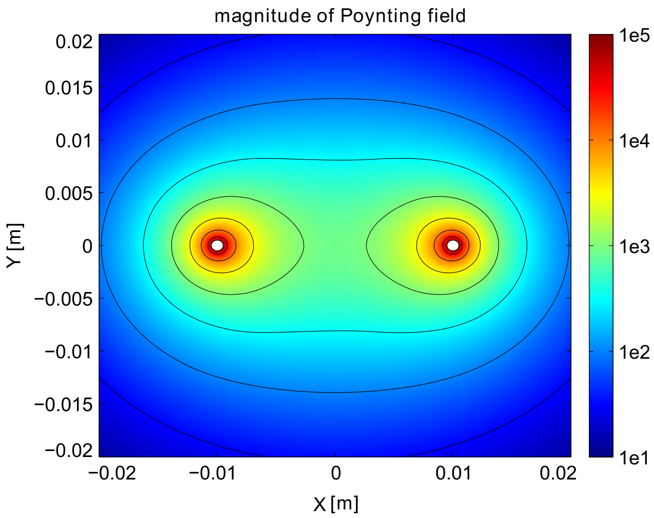

Fig. 4. Plot of $-S_z(x, y)$.

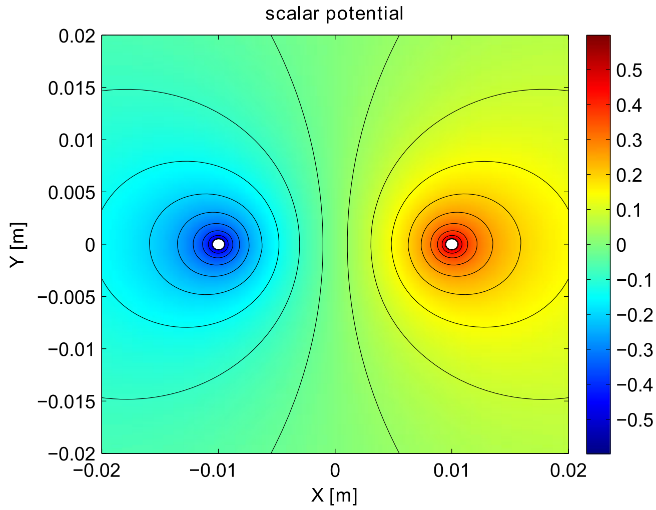

Fig. 7. Plot of $\phi(x, y)$.

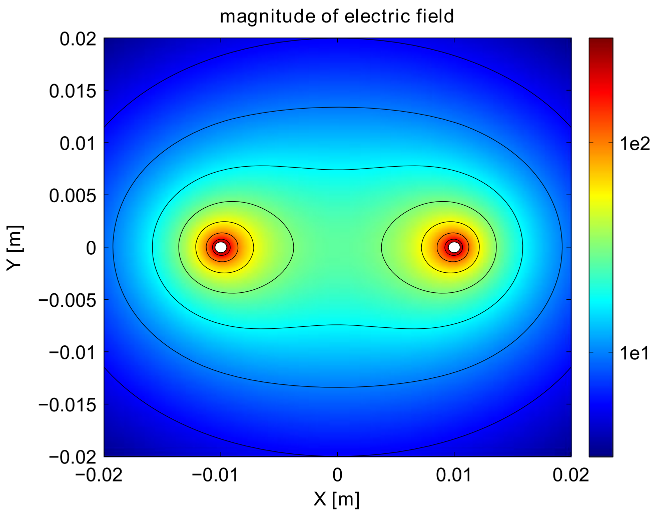

Fig. 5. Plot of $\sqrt{E_x^2 + E_y^2}$.

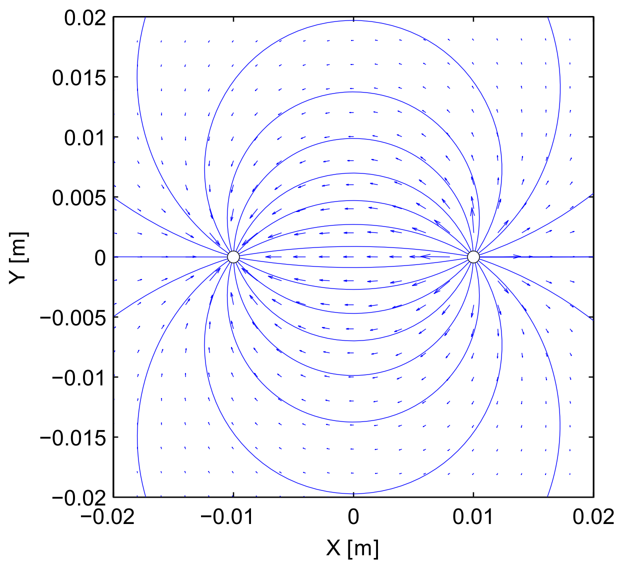

Fig. 8. Vector and fieldline plot of $\boldsymbol{E}(x, y)$.

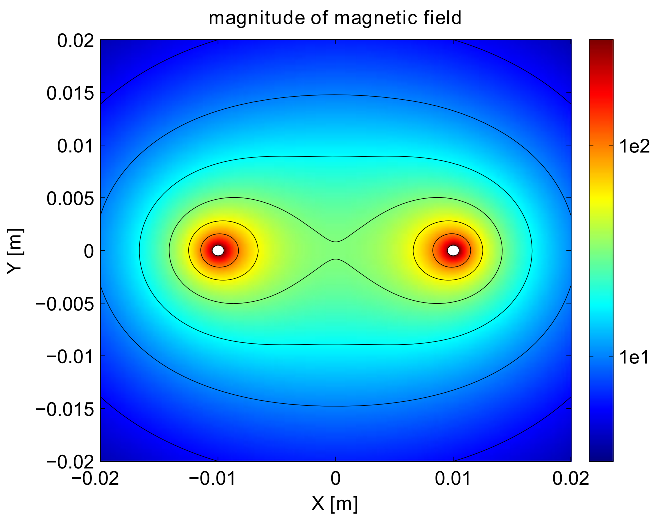

Fig. 6. Plot of $\sqrt{H_x^2 + H_y^2}$.

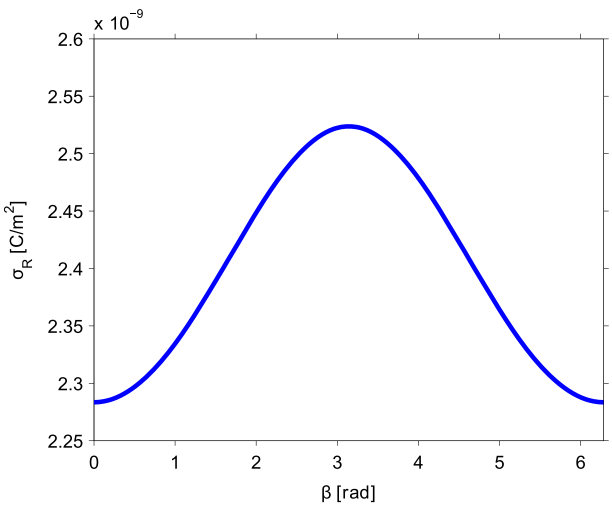

Fig. 9. Surface charge density σ_R on the right wire.

(Fig. 7) outside the wires. Also, some fieldlines (Fig. 8) of the electric field, starting at the right wire, are shown, and the surface charge density on the surface of the right wire (Fig. 9).

As a check, the Poynting vector was numerically integrated over the xy-plane. We obtained 1.004 W, corresponding well with the theoretical value of $1\ \mathrm{V} \times 1\ \mathrm{A} = 1\ \mathrm{W}$.

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