Eur. Phys. J. Appl. Phys. **32**, 159–165 (2005)
DOI: 10.1051/epjap:2005087

THE EUROPEAN
PHYSICAL JOURNAL
APPLIED PHYSICS

Symmetry properties of 2D magnetic photonic crystals with square lattice

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Received: 20 April 2005 / Received in final form: 7 June 2005 / Accepted: 7 July 2005
Published online: 6 December 2005 – © EDP Sciences

Abstract. We consider possible magnetic symmetries of two-dimensional square lattices with circular ferrite rods magnetized by a uniform dc magnetic field. These structures can be used as tunable and nonreciprocal photonic crystals. Classification of eigenmodes in such crystals and compatibility relations are discussed on the basis of magnetic group theory and the theory of (co)representations. Some general electromagnetic properties of the magnetic crystals such as change in the basic domain of the Brillouin zone, change of symmetry in limiting cases, bidirectionality and nonreciprocity, symmetry relations for the waves and lifting of eigenwave degeneracies by dc magnetic field are also discussed.

PACS. 42.70.Qs Photonic bandgap materials – 78.67.-n Optical properties of low-dimensional, mesoscopic, and nanoscale materials and structures – 75.75.+a Magnetic properties of nanostructures

1 Introduction

The theory of groups was used for investigation of electronic band structure of crystals since 30th of the last century. One of the first pioneering works in this field was [1] which had a significant impact on the solid state physics. Later, the theory of magnetic groups was applied to magnetic electronic crystals, and summary of this theory can be found in [2].

Approximately 15 years ago, the idea of using dielectric periodic structures for controlling the frequencies and directions of electromagnetic wave propagation attracted much attention of scientists around the world. These structures were called photonic band gap materials or photonic (electromagnetic) crystals. For investigation of physical properties of the nonmagnetic photonic crystals, the theory of groups was used by many authors (see for example [3,4]).

In the last years, magnetic photonic crystals were investigated intensively because they allow tunability of some crystal parameters and nonreciprocity (see for example [5,6]). Several new physical effects in magnetic crystals were discovered lately. One example is the effect of "the frozen mode" [20]. Another example is the magneto-optic effect similar to the Faraday effect but for the propagation of waves perpendicular to the dc external magnetic field [21]. Still another example is the effect of opening up additional band gaps due to applied dc magnetic field [9,22].

There is a principal difference between the electronic and electromagnetic waves in crystals. Electronic waves are described by scalar (neglecting electron spin) Schrödinger's equation whilst in the case of electromagnetic waves, one deals with vector Maxwell's equations [8]. The symmetry properties of the scalar and vector quantities are different [19]. Therefore we can not transfer directly the results of the electronic wave symmetry theory to the photonic crystals.

One of the important geometries of photonic crystals is two-dimensional (2D) square crystal lattice. Numerical values of the solutions of Maxwell's equations for 2D square magnetic lattice can be found in [9–12]. Our aim in this paper is to discuss some general properties of these crystals which follow from their symmetry. Notice, that in the case of 1D crystals, some elements of the theory of magnetic groups were used in [20].

The organization of the paper is as follows. Section 2 is devoted to the symmetry description of 2D square magnetic lattices, symmetry of the wave vector in the magnetic lattices and compatibility relations. Also, in this section we discuss briefly four limiting cases of these crystals. Some general symmetry properties of the crystals such as reduction of the vector wave equation to a scalar form, symmetry relations for the waves with the opposite sign of the wave vector $\mathbf{k}$, bidirectionality and nonreciprocity of crystals and lifting of the eigenwave degeneracies by dc magnetic field are discussed in Section 3. Concluding remarks of Section 4 finalize the paper.

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2 Symmetry description of 2D square magnetic lattices

Possible symmetries of magnetic lattices. The analyzed structure is shown in Figure 1a. The uniform in z-direction circular ferrite rods are oriented along the z-axis. They form a square lattice in the plane $x0y$. The permeability of the magnetized ferrite rods is a tensor of the second rank $\bar{\mu}(\mathbf{r})$ and the permittivity is a scalar ϵ. The space between the rods is filled with air. The ferrite is in general lossy. Without dc magnetic field, one can consider the ferrite rods as dielectric ones described by a scalar permeability μ.

The square unit sell of the lattice has the period a in both the x- and the y-direction (Fig. 1a). Therefore, the translational symmetry of the lattice is described by the two elementary lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\mathbf{a}_1 = \begin{pmatrix} \mathrm{a} \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ \mathrm{a} \end{pmatrix}. \tag{1}$$

Thus, the permeability $\bar{\mu}(\mathbf{r})$ and the permittivity $\epsilon(\mathbf{r})$ of the whole structure are periodic functions with respect to translations in the lattice, that is $\bar{\mu}(\mathbf{r}+\mathbf{a}) = \bar{\mu}(\mathbf{r})$ and $\epsilon(\mathbf{r}+\mathbf{a}) = \epsilon(\mathbf{r})$ where $\mathbf{a} = m\mathbf{a}_1 + n\mathbf{a}_2$, m and n are integers.

The unit cell of the 2D lattice shown in Figure 1a is a square possessing the geometrical symmetry C_{4v} (in Schönflies notations [2]). The full symmetry of this non-magnetic crystal includes also the restricted Time reversal operator $\mathcal{T}$ [7] and the product of $\mathcal{T}$ with all the geometrical elements of the group C_{4v}, i.e. the full magnetic group can be denoted as $C_{4v} + \mathcal{T}C_{4v}$.

Magnetization of the crystal by a dc magnetic field changes the symmetry of the crystal. In our case of the 2D square lattice, the group decomposition tree of the group C_{4v} is shown in Figure 2. Using this decomposition tree, we can find all the possible magnetic symmetries of our crystal [13]. In what follows, we shall restrict ourselves by the magnetic structures which are magnetized by a uniform dc magnetic field.

Why can one need to consider different symmetries of the magnetic structure under investigation? In physical experiments, for example, a small deviation of the applied dc magnetic field orientation can lead to qualitative change of the basic domain, of the band structure, of the field structure of eigenwaves, etc. Thus, the knowledge of the possible change of symmetry and the consequences of this due to the change of the orientation of the dc magnetic field can be useful in interpreting the obtained experimental results. Besides, different orientations of the dc magnetic field can lead to different physical effects which can be used in electromagnetic devices.

For an arbitrary orientation of $\mathbf{H}_0$ with respect to the axis z in Figure 1a, the resulting group of symmetry of the magnetic crystal contains only the identity element e. This group is trivial and gives no information about the properties of the crystal. We shall consider 3 nontrivial cases.

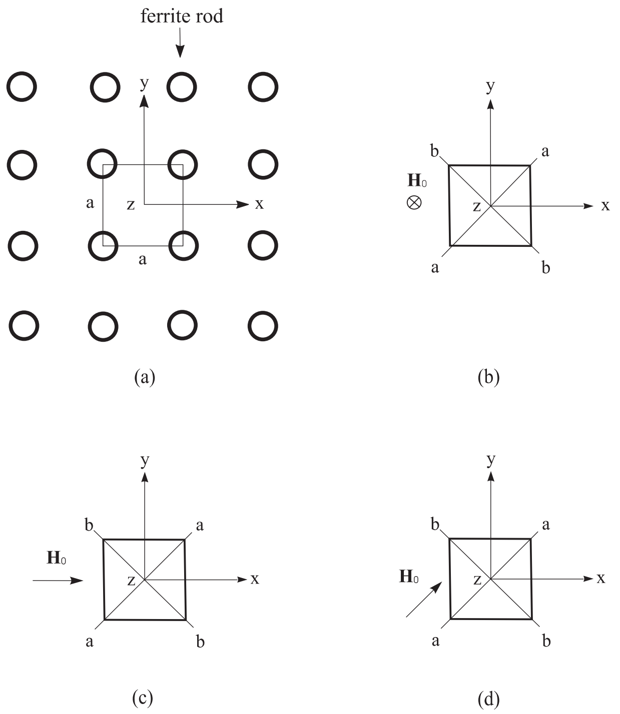

Fig. 1. (a) 2D square lattice of circular cross-section ferrite rods, (b) the unit cell magnetized by $\mathbf{H}_0 \parallel z$, (c) the unit cell magnetized by $\mathbf{H}_0 \parallel x$, (d) the unit cell magnetized by $\mathbf{H}_0 \parallel (a-a)$.

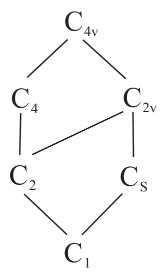

Fig. 2. Subgroup decomposition of the point group C_{4v}.

I. The field $\mathbf{H}_0$ is directed along the axis z (i.e. $\mathbf{H}_0 \parallel z$, Fig. 1b). The resulting group of symmetry of the system "2D square lattice + dc magnetic field" is $C_{4v}(C_4)$.

II. The field $\mathbf{H}_0$ lies in the plane $x0y$. Here, we can consider the following 3 subcases:

a) $\mathbf{H}_0 \parallel x$ (Fig. 1c) or $\mathbf{H}_0 \parallel y$. The cases $\mathbf{H}_0 \parallel x$ and $\mathbf{H}_0 \parallel y$ are physically equivalent.
b) $\mathbf{H}_0 \parallel (a-a)$ (Fig. 1d) or $\mathbf{H}_0 \parallel (b-b)$. The cases $\mathbf{H}_0 \parallel (a-a)$ and $\mathbf{H}_0 \parallel (b-b)$ are also physically equivalent.
In cases IIa and IIb, the resulting group of symmetry is $C_{2v}(C_s)$. From the group-theoretical point of view, the cases IIa and IIb are indistinguishable, because they are described by the same group of symmetry.
c) For any other orientation of $\mathbf{H}_0 \perp z$, the group of symmetry is lower, and it is $C_2(C_1)$.

III. The field $\mathbf{H}_0$ is in one of the planes $x0z$, $y0z$, $(a-a)0z$ or $(b-b)0z$ but it is neither parallel to the axis z

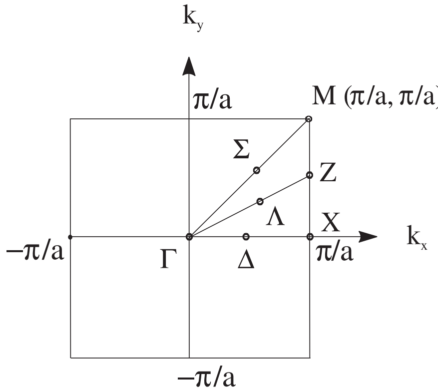

Fig. 3. The reduced Brillouin zone for the 2D square lattice of circular cross-section ferrite rods.

nor perpendicular to it. The group of symmetry in this case is $C_s(C_1)$.

Useful information can be obtained from the irreducible representations (IRREPs) of groups. The IRREPs of the space groups of photonic crystals are defined by $\exp\{i\mathbf{k}\cdot\mathbf{a}\}\cdot\Gamma(\mathcal{R})$ where $\mathbf{k}$ is the wave vector, $\Gamma(\mathcal{R})$ is the rotation-reflection part (including the Time reversal) and $\exp\{i\mathbf{k}\cdot\mathbf{a}\}$ is the translational part of the IRREPs. Our aim is the magnetic point symmetry therefore we shall not consider the irreducible representations of the translational part of the crystal space groups.

Before discussing the symmetry of the wave vector $\mathbf{k}$, we should define the Brillouin zone (BZ) for the above cases of magnetic structures. First of all, the shape of the BZ zone does not coincide in general, with the shape of the unit cell of a lattice [2]. However, in the case of the nonmagnetic square unit cell, the BZ has also the square shape. Besides, in general, a dc magnetic field can change the size and even the shape of the BZ. But in our case of the *uniform* dc magnetic field, the unit cell and consequently, the BZ are not changed because the translational symmetry of the crystal is unchanged after biasing by such a dc magnetic field. Thus, in spite of different magnetic symmetries, the BZ of the photonic crystal with and without magnetization are exactly the same. Therefore, for all our symmetries $C_{4v}(C_4), C_{2v}(C_s)$, $C_2(C_1)$ and $C_s(C_1)$ we shall investigate the square BZ which is identical to the BZ of the nonmagnetic lattice (Fig. 3). The basic domains and the stars for the magnetic crystals with different symmetries are shown in Figure 4. A detailed discussion of the problem of the basic domain in magnetic electronic crystals can be found in [14].

Group of symmetry of the wave vector. The Time reversal operator $\mathcal{T}$ as an element of the group of symmetry of a nonmagnetic crystal, sends $\mathbf{k}$ into $-\mathbf{k}$, i.e.

$$\mathcal{T}\mathbf{k} = -\mathbf{k}. \tag{2}$$

In the cases of magnetic crystals, the Time reversal $\mathcal{T}$ does not exist in "pure" form, but it can enter in the group in the combined operations (a geometrical operation + Time reversal). Let us denote any operator of geometrical symmetry as $\mathcal{R}_1$ and an operator of combined symmetry

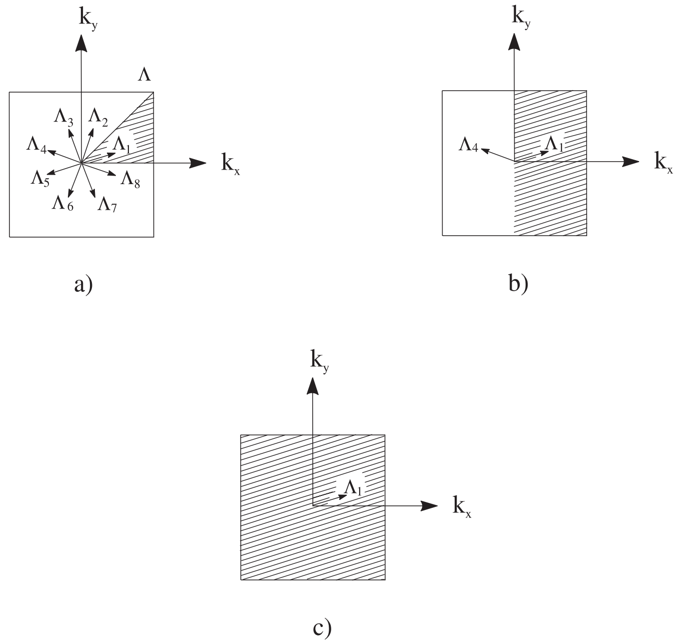

Fig. 4. The basic domains (dashed areas) and the stars for the magnetic crystals with different symmetries: (a) $C_{4v} + \mathcal{T}C_{4v}$ and $C_{4v}(C_4)$, (b) $C_{2v}(C_s)$ and $C_s(C_1)$, (c) $C_2(C_1)$.

as $\mathcal{T}\mathcal{R}_2$. The magnetic little group $\mathbf{M^k}$ consists of those geometrical operators $\mathcal{R}_1$ which transform the wave vector $\mathbf{k}$ into itself or into $\mathbf{k}+\mathbf{G}$ where $\mathbf{G}$ is a reciprocal lattice vector:

$$\mathcal{R}_1\mathbf{k} = \mathbf{k} + \mathbf{G}, \tag{3}$$

and also of the operators $\mathcal{T}\mathcal{R}_2$ with $\mathcal{R}_2$ which transform $\mathbf{k}$ into $-\mathbf{k}$ or into $-\mathbf{k}+\mathbf{G}$:

$$\mathcal{R}_2\mathbf{k} = -\mathbf{k} + \mathbf{G}. \tag{4}$$

The magnetization $\mathbf{H}_0 \parallel z$ reduces the symmetry of the crystal from $C_{4v} + \mathcal{T}C_{4v}$ to $C_{4v}(C_4)$. For the points Γ and M of the BZ (Fig. 3), the wave vectors have the symmetry $C_{4v}(C_4)$ (Tab. 1). The symmetry of the point X is $C_{2v}(C_2)$. The symmetry of the vectors Δ and Z is $C_s(C_1)$. The wave vector Λ in a general point of the BZ has no symmetry. Analogously, we can define symmetry of the point and lines of the BZ for other orientations of dc magnetic field.

Compatibility relations. For magnetic crystals, we can consider two types of compatibility relations:

(1) between the corepresentations of the groups for the crystal in nonmagnetic and magnetic state with different magnetic symmetries (i.e. with different orientation of dc magnetic field);
(2) for a given magnetic symmetry, between the corepresentations of $\mathbf{k}$ vectors at special points and lines of the BZ.

An example of the first type of compatibility relations is given in Table 2 (for representations, we use in this paper the notations of [2]). The letter D in front of A, B, etc.

Table 1. Little groups and their elements for points and lines of symmetry for square magnetic lattice with dc magnetic field $\mathbf{H}_0 \parallel z$.

Symmetry symbol	Representative wave vector $\mathbf{k}$	Little group	Order of the group	Elements of the group
Γ	$\pi/a(0,0)$	$C_{4v}(C_4)$	8	e, C_4, C_4^{-1}, C_2, $\mathcal{T}\sigma_x$, $\mathcal{T}\sigma_y$, $\mathcal{T}\sigma_{(a-a)}$, $\mathcal{T}\sigma_{(b-b)}$
M	$\pi/a(1,1)$	$C_{4v}(C_4)$	8	e, C_4, C_4^{-1}, C_2, $\mathcal{T}\sigma_x$, $\mathcal{T}\sigma_y$, $\mathcal{T}\sigma_{(a-a)}$, $\mathcal{T}\sigma_{(b-b)}$
X	$\pi/a(1,0)$	$C_{2v}(C_2)$	4	e, C_2, $\mathcal{T}\sigma_x$, $\mathcal{T}\sigma_y$
Δ	$\pi/a(\alpha,0)$	$C_s(C_1)$	2	e, $\mathcal{T}\sigma_x$
Z	$\pi/a(1,\beta)$	$C_s(C_1)$	2	e, $\mathcal{T}\sigma_y$
Λ	$\pi/a(\alpha,\beta)$	C_1	1	e

Table 2. Compatibility between the groups $C_{4v} + \mathcal{T}C_{4v}$ and $C_{4v}(C_4)$.

$C_{4v} + \mathcal{T}C_{4v}$	DA_1	DA_2	DB_1	DB_2	DE
$C_{4v}(C_4)$	DA	DA	DB	DB	$D\,^1E + D\,^2E$

in this table is used to denote the corepresentations of the magnetic groups.

In order to define the compatibility relations between the corepresentations of $\mathbf{k}$ vectors at the special points and lines of the BZ of a magnetic crystal with the symmetry $G(H)$, it suffices to use only the representations of the unitary subgroup H of the group $G(H)$ [17]. This simplifies the procedure of constructing the compatibility tables.

Limiting cases. One can consider 4 limiting cases of our problem. The first one is the *material* limiting case. When the dc magnetic field goes to zero, the permeability tensor (the material parameter) of the ferrite rods reduces to a scalar. Besides, two *geometric* limiting cases can be considered also. Directing the fractal volume occupied by the ferrite to 1, one comes to the limit of the homogeneous ferrite. On the opposite extreme of the zero fractal ferrite volume, one has the homogeneous dielectric medium. The discrete symmetry (periodicity) of the crystal is transformed in both cases in continuous symmetry of the homogeneous media.

Still another limiting case is the long-wavelength approximation (i.e. for $\mathbf{k} \to 0$). In this approximation, the photonic crystal behaves like a homogeneous media. The resulting symmetry of the medium is defined by symmetry of the ferrite rods, of the crystal lattice and of the applied field $\mathbf{H}_0$.

3 Symmetry properties of eigenwaves

3.1 Wave equations

We shall consider Maxwell's equations in the frequency domain. The vector wave equations for the electric displacement $\mathbf{D}(\mathbf{r})$ and the magnetic induction $\mathbf{B}(\mathbf{r})$ in the photonic ferrite crystal with the tensor $\bar{\mu}(\mathbf{r})$ and the scalar $\epsilon(\mathbf{r})$ have the following form:

$$\mathcal{L}_D\mathbf{D}(\mathbf{r}) = \frac{\omega^2}{c^2}\mathbf{D}(\mathbf{r}), \tag{5}$$

$$\mathcal{L}_B\mathbf{B}(\mathbf{r}) = \frac{\omega^2}{c^2}\mathbf{B}(\mathbf{r}), \tag{6}$$

where the differential operators $\mathcal{L}_D$ and $\mathcal{L}_B$ are

$$\mathcal{L}_D = \nabla \times \left\{\bar{\mu}^{-1}(\mathbf{r})\nabla \times \left[\epsilon^{-1}(\mathbf{r})\cdot \quad\right]\right\}, \tag{7}$$

$$\mathcal{L}_B = \nabla \times \left\{\epsilon^{-1}(\mathbf{r})\nabla \times \left[\bar{\mu}^{-1}(\mathbf{r})\cdot \quad\right]\right\}, \tag{8}$$

$\bar{\mu}^{-1}(\mathbf{r})$ is the tensor inverse to $\bar{\mu}(\mathbf{r})$, c is the light velocity in vacuum. Equations (5 and 6) should be solved together with the equations

$$\nabla \cdot \mathbf{D} = 0, \tag{9}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{10}$$

and the fields $\mathbf{E}$ and $\mathbf{H}$ can be found using the constitutive relations

$$\mathbf{D} = \epsilon(\mathbf{r})\mathbf{E}, \quad \mathbf{B} = \bar{\mu}(\mathbf{r})\mathbf{H}. \tag{11}$$

The Space and Time reversal symmetry properties of the vector $\mathbf{D}$ and $\mathbf{E}$ are the same. They are even in Time polar vectors connected by the scalar $\epsilon(\mathbf{r})$. The Space and Time reversal symmetry properties of the vector $\mathbf{B}$ and $\mathbf{H}$ are also the same. They are odd in Time axial vectors related by the tensor $\bar{\mu}(\mathbf{r})$. The tensor $\bar{\mu}(\mathbf{r})$ has the symmetry of the magnetized ferrite medium. Thus, the symmetry properties of the crystals can be discussed in terms of $\mathbf{D}$ and $\mathbf{B}$ vectors or equivalently, in terms of $\mathbf{E}$ and $\mathbf{H}$ vectors.

Following [15], we write the vector $\mathbf{B}$ as a plane wave

$$\mathbf{B} = e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{u}_{\mathbf{k}n}(\mathbf{r}), \tag{12}$$

where $\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$ is a periodic function with the period $\mathbf{a}$, n is a band index. The quantity $\exp(i\mathbf{k}\cdot\mathbf{r})$ changes the sign of its exponent under Time reversal (this corresponds to changing the direction of propagation). Taking into account this circumstance, we can discuss the symmetry properties of the vector $\mathbf{B}$ in terms of the vector $\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$.

Substituting $\mathbf{B}$ in (6) by expression (12), we obtain the eigenvalue equation which contains the wave vector $\mathbf{k}$:

$$(i\mathbf{k}+\nabla)\times\left\{\epsilon^{-1}(\mathbf{r})(i\mathbf{k}+\nabla)\times\left[\bar{\mu}^{-1}(\mathbf{r})\cdot\mathbf{u}_{\mathbf{k}n}(\mathbf{r})\right]\right\} = \frac{\omega_n^2}{c^2}\mathbf{u}_{\mathbf{k}n}(\mathbf{r}), \tag{13}$$

or

$$\mathcal{L}_u\mathbf{u}_{\mathbf{k}n}(\mathbf{r}) = \frac{\omega_n^2}{c^2}\mathbf{u}_{\mathbf{k}n}(\mathbf{r}). \tag{14}$$

Analogously, writing the vector $\mathbf{D}$ as

$$\mathbf{D} = e^{i(\mathbf{k}\cdot\mathbf{r})}\mathbf{v}_{\mathbf{k}n}(\mathbf{r}), \tag{15}$$

we can obtain the following wave equation:

$$(i\mathbf{k}+\nabla)\times\left\{\bar{\mu}^{-1}(\mathbf{r})\cdot(i\mathbf{k}+\nabla)\times\left[\epsilon^{-1}(\mathbf{r})\mathbf{v}_{\mathbf{k}n}(\mathbf{r})\right]\right\} = \frac{\omega_n^2}{c^2}\mathbf{v}_{\mathbf{k}n}(\mathbf{r}), \tag{16}$$

or

$$\mathcal{L}_v\mathbf{v}_{\mathbf{k}n}(\mathbf{r}) = \frac{\omega_n^2}{c^2}\mathbf{v}_{\mathbf{k}n}(\mathbf{r}). \tag{17}$$

The symmetry properties of the vector $\mathbf{D}$ can be discussed in terms of the vector $\mathbf{v}_{\mathbf{k}n}(\mathbf{r})$ taking into account the above mentioned property of the exponent $e^{i(\mathbf{k}\cdot\mathbf{r})}$.

3.2 Reduction of the vector wave equations to a scalar form

We consider propagation of waves in the plane $x0y$. For this problem in 2D *nonmagnetic* square lattices, the general vector wave equation can be reduced to two independent scalar equations, for E-polarized modes with the components E_z, H_x, H_y and for H-polarized modes with the components H_z, E_x, E_y [4]. This property of the wave equation is a consequence of symmetry. Namely, it is stipulated by the presence of the symmetry plane σ_z [15]. Notice that σ_z is not an element of the groups of symmetry of our idealized 2D problem. It is an element of the corresponding 3D problem. But the real problem is always 3D, therefore we can use σ_z in the investigation of the nonmagnetic lattice as well.

In a homogeneous infinite ferrite medium, the reduction of the vector wave equation to two independent scalar differential equations of the second order is possible for two particular cases of the wave vector orientation with respect to the dc magnetic field: when $\mathbf{k}$ is parallel to the dc magnetic field $\mathbf{H}_0$ and $\mathbf{k}$ is perpendicular to $\mathbf{H}_0$ [16]. For the magnetic photonic crystals, the reduction of the vector wave equation to 2 scalar equations is possible for the orientation of the dc magnetic field parallel to the axis z, i.e. for $\mathbf{H}_0 \parallel z$, because the plane of symmetry σ_z in this case exists as well. But if $\mathbf{H}_0 \parallel x0y$, the plane σ_z is transformed into the antiplane $\mathcal{T}\sigma_z$. A reduction of the vector wave equation to 2 scalar equations in this case is impossible. The operator $\mathcal{T}\sigma_z$ transforms $+\mathbf{k}$ into $-\mathbf{k}$, therefore the presence of this operator defines the relation $\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k})$, i.e. bidirectionality for any direction of $\mathbf{k}$ in the plane $x0y$. Bidirectionality is discussed below in Section 3.4.

3.3 Symmetry of eigenmodes

It is well known [2] that the dispersion characteristics $\omega_n(\mathbf{k})$ of the magnetic crystal have the full symmetry of the point magnetic group of the crystal. Thus, we can use the IRREPs of the magnetic little groups to classify the eigenmodes. In fact, in most cases for the correct classification of the eigenmodes, it is sufficient to use only the unitary subgroups of the corresponding magnetic little groups. The peculiarities of the IRREPs of the point magnetic groups are discussed in [2].

Using the elements of magnetic little groups one can find some restrictions on the field structure of eigenwaves. One example of such restrictions was discussed in Section 3.2.

The symmetry elements of the crystal magnetic group which change the sign of $\mathbf{k}$ can be used to relate the structure of electromagnetic waves propagating in the opposite directions. Let us consider the case of $\mathbf{H}_0 \parallel x0y$. If an eigenwave corresponding to the branch $\omega_n(\mathbf{k})$ has the fields $\mathbf{v}_{\mathbf{k}n}(\mathbf{r})$ and $\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$, the eigenwave corresponding to the branch $\omega_m(-\mathbf{k})$ can be found applying the operator $\mathcal{T}\sigma_z$ to the fields $\mathbf{v}_{\mathbf{k}n}(\mathbf{r})$ and $\mathbf{u}_{\mathbf{k}n}(\mathbf{r})$.

3.4 Bidirectionality and nonreciprocity of crystals

The problem of bidirectionality of electromagnetic waves in homogeneous media is discussed in [13] and in electromagnetic waveguides in [18]. We can also apply the notion of bidirectionality to photonic crystals. We call a given magnetic photonic crystal bidirectional for a given direction $\mathbf{k}$ if there is an operator $\mathcal{R}_3$ or $\mathcal{T}\mathcal{R}_4$ such that

$$\mathcal{R}_3\mathbf{k} = -\mathbf{k}. \tag{18}$$

or

$$\mathcal{T}\mathcal{R}_4\mathbf{k} = -\mathbf{k}. \tag{19}$$

With this condition, for any branch of the dispersion characteristic $\omega_n(\mathbf{k})$ with the vector $\mathbf{k}$ there exists another branch $\omega_m(-\mathbf{k})$ with the vector $-\mathbf{k}$ such that

$$\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k}). \tag{20}$$

We use in (20) different subindexes n and m, because, in general, the structure of the electromagnetic field of eigenwaves with $\mathbf{k}$ and $-\mathbf{k}$ is different.

The following symmetry elements change the sign of the vector **k** defining bidirectionality in our photonic crystals:

- reflection in a plane for the direction of propagation normal to the plane, i.e. σ_x, σ_y, $\sigma_{(a-a)}$, $\sigma_{(b-b)}$,
- rotation about an axis through π for the directions of propagation perpendicular to this axis, i.e. C_2,
- reflection in an antiplane for the directions of propagation parallel to the plane, i.e. $\mathcal{T}\sigma_x$, $\mathcal{T}\sigma_y$, $\mathcal{T}\sigma_{(a-a)}$, $\mathcal{T}\sigma_{(b-b)}$, and also $\mathcal{T}\sigma_z$.

In a nonmagnetic crystal, the notion of bidirectionality is related to the notion of equivalent directions. All the physical properties of the crystal along the equivalent directions (not necessarily opposite) are the same. This is stipulated by the presence of some elements of symmetry: axes, planes and the center. However, existence of these symmetry elements in magnetic crystals does not always lead to equivalence of the directions. For example, a plane of symmetry in a nonmagnetic crystal defines equivalent directions normal to the plane. But the plane of symmetry which is perpendicular to a dc magnetic field does not define equivalent directions. For the opposite directions normal to this plane, the circularly polarized eigenwaves of the same handedness have different propagation constants, and this defines the well-known nonreciprocal Faraday effect.

The dispersion characteristics of a bidirectional crystal for a given direction **k** are symmetric with respect to the sign of **k**. However, it does not always mean that the crystal is reciprocal. The relation $\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k})$ is not a sufficient condition for nonreciprocity, that is the bidirectional crystal can be both reciprocal and nonreciprocal.

Now, let us apply to the problem of reciprocity of the crystals. Different definitions of reciprocity are used in electromagnetic theory. We adopt here the notion of reciprocity related to the restricted Time reversal symmetry [7]. The presence of the Time reversal $\mathcal{T}$ in the group $C_{4v} + \mathcal{T}C_{4v}$ for the nonmagnetic crystal leads to reciprocity and to a general relation $\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k})$ for dispersion characteristics for any direction **k**. This symmetry of the dispersion characteristics allows one to reduce the band calculations by half.

This is not the case for magnetic crystals because the Time reversal is not present in the magnetic little group. The nonreciprocity of a magnetic crystal can manifest itself in difference of the wave vectors (phase and velocity difference), electromagnetic field structure (in particular, polarization) or amplitude for the waves propagating in opposite directions. For magnetic crystals, in general, the dispersion characteristics are different for **k** and $-\mathbf{k}$, i.e. $\omega_n(\mathbf{k}) \neq \omega_m(-\mathbf{k})$ and one has to calculate both $\omega_n(\mathbf{k})$ and $\omega_m(-\mathbf{k})$. But for some directions in magnetic crystals, we can have $\omega_n(\mathbf{k}) = \omega_m(-\mathbf{k})$ and this is the case of bidirectionality discussed above.

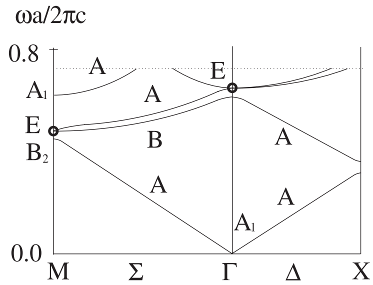

Fig. 5. Dispersion diagrams for nonmagnetic crystal with with symmetry C_{4v} for E-polarization, adapted from [4]. The points marked by the circles correspond to the doubly degenerate representation E which can split by dc magnetic field.

3.5 Lifting of degeneracy by dc magnetic field

In Figure 5, the dispersion diagram for the nonmagnetic crystal for the E polarization is shown [4]. The points Γ and M are denoted by circles. With dc magnetic field applied to the crystal, we can expect these eigenwaves with symmetry-induced degeneracies to be split by symmetry reduction into 2 different nondegenerate eigenwaves (see Tab. 2). Notice that the effect of degeneracy lifting can not be investigated in the simplified description of the magnetic (or semiconductor) 2D crystals with the tensor $\bar{\mu}(\mathbf{r})$ (or $\bar{\epsilon}(\mathbf{r})$) which does not contain off-diagonal elements (this approximation was used for example, in [10,12]).

4 Conclusions

We have discussed in this paper some general properties of 2D magnetic photonic crystals with square lattice with circular ferrite elements using symmetry arguments. The group theoretical methods are also valid for crystals with any geometry of the ferrite elements and for nonuniform magnetization. One should only remember that the resulting group of crystal symmetry will depend on the shape of the ferrite elements and on the geometry of external dc magnetic field.

Notice that in some cases of magnetic crystals, an additional degeneracy of eigenmodes can arise which is not predicted by the unitary subgroup, but this effect does not occur in the magnetic crystals with *uniform* magnetization which is the case of our present investigation.

With small modifications, the symmetry analysis of this paper can be applied also to the photonic crystals with magnetized semiconductor elements.

This work was supported by Brazilian agency CNPq.

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