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Group theoretical approach to complex and bianisotropic media description*

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Abstract. The key idea of this work is application of the concept of magnetic groups and Curie's principle to the tensor description of complex and bianisotropic media. It is shown how to find the symmetry group of the media under external perturbation and to calculate the structure of the constitutive tensors starting from the symmetry of the medium. The theory may be used as a first step in the problem of new artificial material synthesis.

PACS. 81.05.Zx New materials: theory, design, and fabrication – 11.30.Er Charge conjugation, parity, time reversal, and other discrete symmetries

1 Introduction

Fast development of the theory and electromagnetic applications of artificial complex and bianisotropic media leads to necessity of investigation of general properties of such media. Some general properties of linear media are defined by space-time symmetry and may be derived using group-theoretical approach.

Symmetry may be applied to the analysis of media at different levels of medium description. For example in the case of inhomogeneous symmetrical media with regular structure we may use the space magnetic groups which take into account periodicity of the media. These groups involve lattice translation operations, rotation-reflection ones and their combinations, and also combinations of the above operations with time reversal. When multipole terms are used in the constitutive relations [1], symmetry principles and methods may also give some general information. In this paper, we will restrict ourselves by consideration of homogeneous medium approximation. A mathematical tool of our investigation is magnetic point groups.

In the theory of em field, the em properties of the media are often described in terms of constitutive parameters. One of the possible forms of the constitutive relations for time harmonic field is [2]:

$$\mathbf{D} = [\varepsilon]\mathbf{E} + [\xi]\mathbf{H}, \tag{1}$$

$$\mathbf{B} = [\zeta]\mathbf{E} + [\mu]\mathbf{H}. \tag{2}$$

The parameters of the constitutive tensors $[\mu]$, $[\varepsilon]$, $[\xi]$ and $[\zeta]$ are defined by different methods, for example, by methods of quantum physical kinetics, by electrodynamic calculations or by experimentation.

The four 3×3 tensors of equations (1, 2) in the most general form contain 36 independent parameters. Under some conditions, the number of independent parameters may be reduced. For example, the Post's uniformity constraint [3,4] which is an algebraic relation between the constitutive parameters, reduces the number of independent parameters to $36 - 1 = 35$. The constitutive tensors for the reciprocal media satisfying the conditions

$$[\mu] = [\mu]^t, \quad [\varepsilon] = [\varepsilon]^t, \quad [\xi] = -[\zeta]^t \tag{3}$$

where t stands for transposition, contain 21 material parameters [2]. The conditions (3) are the manifestation of time symmetry of the media (see Appendix).

Lossless media are described by the conditions [2]

$$[\mu] = [\mu]^\dagger, \quad [\varepsilon] = [\varepsilon]^\dagger, \quad [\xi] = [\zeta]^\dagger \tag{4}$$

where $\dagger$ denotes a conjugate and a transpose of the tensor. The constitutive tensors of the lossless media contain 21 independent complex parameters. Sometimes the equations (4) are also considered as special symmetry conditions [2].

Space and space-time symmetry of media also leads to reduction of the number of independent parameters. We shall consider in this paper three types of symmetry transformations (mappings):

(1) space transformations (rotations and reflections);
(2) time transformation (time reversal operation, which takes t into $-t$);
(3) space-time transformations (rotations and reflections combined with time inversion).

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The first and the second transformations are used in order to find the structure of constitutive tensors for nonmagnetic media. For magnetic media, we have to utilize the first and sometimes the third type of symmetry transformations.

The problems of spatial mapping and time reversal of electromagnetic field have been considered by different authors, for example by Jackson [5], Birss [6], Baum and Kritikos [7]. A detailed description of the problem is given in the papers of Altman, Schatzberg and Suchy [8,9]. They used the concept of formally adjointness of Maxwell equations which means transposing all the operators and changing the sign of the differential operators. It corresponds to temporal and spatial reflection transformation $\partial/\partial t \to -\partial/\partial t$ and $\nabla \to -\nabla$ [10]. As a result, the Maxwell's equations are converted into adjoint ones which correspond to nonphysical electromagnetic field but nevertheless this permits the solution of the mode orthogonality and reciprocity problems.

In order to describe the constitutive tensor transformation properties, we will use in this work a different approach based on magnetic group theory and on Onsager's principle. The unitary and antiunitary symmetry operators of the magnetic group describing a medium will be applied directly to the constitutive relations (see Appendix). The properties of the unitary operators and the antiunitary time reversal operator give the transformation laws for the constitutive tensors. The magnetic group of symmetry of complex and bianisotropic media which may be under external perturbation is defined by Curie's principle [11].

Thus, the structure of the constitutive tensors of complex and bianisotropic media is defined in many respects by symmetry of the media and of external perturbation. The dynamical peculiarities of the media are reflected in the numerical values of the constitutive parameters and sometimes in a simplification of the tensor structure in comparison with that calculated by symmetry methods. It is illustrated by two examples: we calculate the constitutive tensors of the media described by the continuous magnetic groups of the third category and of moving media. The presented approach also allows one to make a first step in the problem of new artificial medium synthesis.

2 Brief description of magnetic groups

In the nonmagnetic state, a medium is described by magnetic group of symmetry of the first category G [12]. The group G consists of the unitary subgroup H (it contains the usual rotation-reflection operations) and products of the Wigner time reversal operator T [21] with all the elements of H. The full magnetic group of the media is then $G = H + TH$ including $T = Te$ where e is the unit element. Such media are reciprocal.

In the case of magnetic groups of the second category, there is no elements with time reversal operator T, and T is not an element of the groups, therefore the media are in general nonreciprocal.

The magnetic groups of the third category contain in addition to the unitary operators of geometric symmetry, an equal number of antiunitary operators which are the product of the usual geometrical symmetry operators and the operator T. These combined operators lead to the existence of antiaxes, antiplanes and anticenter of symmetry. Notice that the operator T itself is not an element of the magnetic groups of the third category, therefore the media described by these magnetic groups are also in general nonreciprocal.

3 Calculation of the constitutive tensor

An algorithm of determination of the tensor structure is as follows:

1. determination of the group of symmetry of the medium (here we can use Curie's principle of symmetry superposition [11] which defines the resultant symmetry of a complex object);
2. determination of the generators of the group [13] (generators are a small number of group elements which define all the group elements);
3. calculation of the constitutive tensors, using equations (A.19–A.22).

The magnetic group of symmetry of a medium is defined by the symmetry of the constitutive particles, their mutual arrangement, the symmetry of the host medium, the symmetry of the external perturbations. For example, bianisotropic materials for electromagnetic applications which have been suggested by Kamenetskii [14], consist of magnetostatic wave ferrite resonators with surface metalization. Different forms of the metalization and different structures and orientations of dc magnetic field lead to a great diversity of the magnetic groups which describing such media [15].

In the problem of determination of the symmetry group we must take into account the following. In many cases natural and artificial media may be considered as isotropic, when external fields and forces (perturbations) are not applied. External perturbations for example, electric and magnetic fields, mechanical forces, temperature fields and their combinations, change some properties of the media. In particular, a medium under perturbation may become anisotropic or gyrotropic one. The external perturbations may be both being created deliberately (for example, magnetization of ferrites by dc magnetic field), and undesirable (nonuniform heating in powerful *em* devices).

In the absence of external perturbations, the isotropic homogeneous media are described by two point continuous symmetry groups, namely K_h and K. The group K_h is the highest possible one and it corresponds to the symmetry of sphere with the planes of symmetry. The group K, on the other hand, may be represented by a sphere without planes of symmetry with all the diameters twisted through an equal angle in one direction. The isotropic chiral medium has such a symmetry.

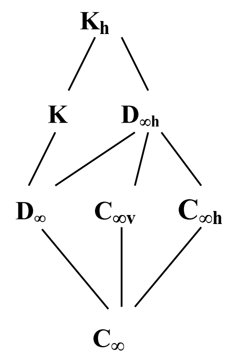

Fig. 1. Group decomposition of continuous groups.

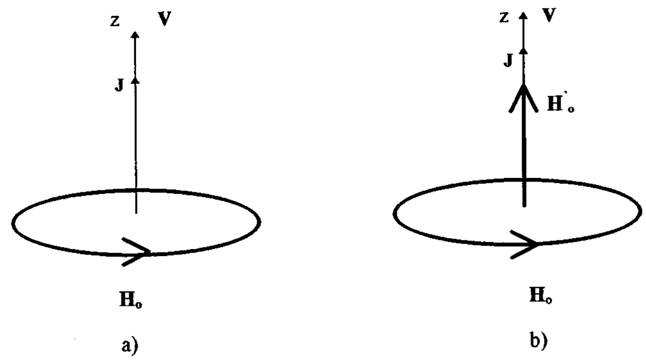

Fig. 2. Structure of dc magnetic field describing symmetry of moving media: (a) for dielectric media, (b) for gyrotropic media.

External perturbations may change the initial symmetry K_h and K. These perturbations may be described by the tensors of the second, first and zero rank (see Appendix). Scalar perturbations obviously can not change the initial symmetry of the isotropic medium. Several examples of vector perturbations are as follows:

- the vector **E** of the uniform electric field is a polar one and it is described by the symmetry $C_{\infty\mathrm{v}}$ (we use in this paper Schoenflies notations of the point groups [12]). The group $C_{\infty\mathrm{v}}$ has one axis of symmetry of the ∞-order and infinite number of the planes of symmetry coinciding with this axis;
- the vector **H** of uniform dc magnetic field is an axial one and it has the magnetic symmetry $D_{\infty\mathrm{h}}$ $(C_{\infty\mathrm{h}})$. This group contains one ∞-fold axis, the plane of symmetry perpendicular to the axis and infinite number of antiaxis lying in the plane.

A medium with the symmetry K_h under an external perturbation acquires the symmetry of this perturbation. The symmetry of K-medium under an external perturbation may be found using Curie's principle of symmetry superposition. For example, the isotropic chiral medium with applied dc magnetic field acquires the symmetry D_∞ (C_∞), because the group K describing chiral medium and the group $D_{\infty\mathrm{h}}$ $(C_{\infty\mathrm{h}})$ describing dc magnetic field have one common element, namely the axis C_∞. Besides, an infinite number of the axes of the second order which are perpendicular to the axis C_∞ are converted under dc magnetic field into the antiaxes.

4 Media described by the continuous magnetic groups of the third category

The continuous groups are those which contain at least one axis of symmetry of an infinite order C_∞. The group decomposition of the continuous groups is shown in Figure 1. The groups of the third category are [16]:

$$K_\mathrm{h}(K),\ D_{\infty\mathrm{h}}\ (D_\infty),\ D_{\infty\mathrm{h}}\ (C_{\infty\mathrm{v}}),\ D_{\infty\mathrm{h}}\ (C_{\infty\mathrm{h}}),\ D_\infty\ (C_\infty),\ C_{\infty\mathrm{v}}\ (C_\infty),\ C_{\infty\mathrm{h}}\ (C_\infty). \quad (5)$$

Using the relations (A.19–A.22) we have calculated the constitutive tensors for these groups which are in Table 1.

One should emphasize that Table 1 contains the complete classification of the symmetrical media described by the continuous magnetic groups of the third category.

We see in Table 1 that the tensors $[\zeta]$ are expressed in terms of the $[\xi]$-elements. It is explained by the use of the relations (A.22) for their calculations, *i.e.* by the presence of the antiunitary elements in the groups of the third category. Physically, it is a consequence of the Onsager principle [17] because the antiunitary elements contain the time reversal operator.

In Table 1, the Post's uniformity constraints [3,4] on the tensor elements are written as well. These constraints are algebraic relations between the elements of the tensors. It has been found that the existence of a plane of symmetry, an antiaxis of the second order or an improper axis S_4 lead automatically to the fullfilment of the Post's constraint.

The symmetry of the medium and as a result, the symmetry of the constitutive tensors can define some physical properties and possible physical effects in the media, for example, Faraday rotation, optical activity, magnetoelectric effect, etc.

5 Moving media

The constitutive tensors for moving media have been obtained from covariantness of the Maxwell equations under Lorentz transformation [18]. Here, we will define the constitutive tensors for moving dielectric and gyrotropic media using group theoretical approach and compare them with the known ones.

From the physical experiments provided by Eihenwald [19] it is known that a moving dielectric behaves as a permanent magnet, *i.e.* the polarized positive and negative charges of the dielectric form linear electric currents **J** (Fig. 2a, **v** is the velocity of the moving medium in the z-direction). The symmetry of the magnetic field which they produce is $D_{\infty\mathrm{h}}$ $(C_{\infty\mathrm{v}})$. The calculated constitutive tensors of this group are given in Table 1. The structure of these tensors coincide with that obtained by an electrodynamical method in [18]. The dynamical peculiarities

Table 1. The constitutive tensors for the media described by the continuous groups of the third category.

N°	Magnetic group of third category	$[\mu]$	$[\varepsilon]$	$[\xi]$	$[\zeta]$	Number of independent parameters$^{(*)}$
1	K_{h} (K)	μ	ε	ξ	ξ	2
2	$D_{\infty\mathrm{h}}$ (D_∞)	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	5
3	$D_{\infty\mathrm{h}}$ $(C_{\infty\mathrm{v}})$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 & \xi_{12} & 0 \\ -\xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & -\xi_{12} & 0 \\ \xi_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$	5
4	$D_{\infty\mathrm{h}}$ $(C_{\infty\mathrm{h}})$	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	$\begin{vmatrix} 0 \end{vmatrix}$	6
5	D_∞ (C_∞)	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} -\xi_{11} & -\xi_{12} & 0 \\ \xi_{12} & -\xi_{11} & 0 \\ 0 & 0 & -\xi_{33} \end{vmatrix}$	9
6	$C_{\infty\mathrm{v}}$ (C_∞)	$\begin{vmatrix} \mu_{11} & \mu_{12} & 0 \\ -\mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ -\varepsilon_{12} & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	8
7	$C_{\infty\mathrm{h}}(C_\infty)$	$\begin{vmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{vmatrix}$	$\begin{vmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & \xi_{12} & 0 \\ -\xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	$\begin{vmatrix} \xi_{11} & -\xi_{12} & 0 \\ \xi_{12} & \xi_{11} & 0 \\ 0 & 0 & \xi_{33} \end{vmatrix}$	6

(*) The number of independent parameters is defined taking into account the Post's constraint [3,4]:

(1) $\xi = 0$;

(2) $2\xi_{11}/\mu_{11} + \xi_{33}/\mu_{33} = 0$;

(3) the Post's constraint is fulfilled because there exists a plane of symmetry C_{s};

(4) it is fulfilled because $[\xi] = [\zeta] \equiv 0$;

(5) it is fulfilled because there exists an antiaxis TC_2;

(6) $2(\mu_{11}\xi_{11} + \mu_{12}\xi_{12})/(\mu_{11}^2 + \mu_{12}^2) + \xi_{33}/\mu_{33} = 0$;

(7) $2\xi_{11}/\mu_{11} + \xi_{33}/\mu_{33} = 0$.

From the tensors $[\xi]$ and $[\zeta]$ of case 7, one can obtain the corresponding tensors of 2, 3 and 4 as a special case. Case 2 is a special one of 6, case 3 is a special one of 5.

of the moving medium lead to an additional constraint on the tensor elements $\varepsilon_{11}\mu_{33} = \varepsilon_{33}\mu_{11}$. Thus, the moving dielectric medium is a special case of the media described by the magnetic group $D_{\infty\mathrm{h}}$ $(C_{\infty\mathrm{v}})$.

For the moving gyrotropic media, we may consider the magnetic symmetry as follows. As in the case above in any moving media there exists a ring magnetic field $\mathbf{H}_0$ produced by moving charges. Besides, there is a uniform dc magnetic field $\mathbf{H}_0'$ which gives the gyrotropic effect (Fig. 2b). In accord with Curie's principle, the combination of these two fields is described by the magnetic group D_∞ (C_∞) with an axis of infinite order C_∞ along the z-axis and an infinite number of the antiaxes lying in the plane $z = 0$. The constitutive tensors for this group are written in Table 1. Comparing them with the tensors obtained in [20] we see that they coincide if $\xi_{33} = 0$. Thus, again the concrete dynamical properties of medium simplify the structure of the constitutive tensors obtained from the symmetry consideration, *i.e.* the actual tensors are a special case of those obtained from symmetry.

Notice that one and the same group of symmetry may describe different media. For example, it has been shown in Section 3, that the group D_∞ (C_∞) describes also chiral media with applied dc magnetic field.

6 Discussion of the results and conclusions

Application of Curie's principle and point magnetic groups allows one to reduce significantly the number of independent tensor parameters and to find the structure of the tensors by means of simple calculations. A restriction of the described above geometrical method is impossibility to obtain numerical values of the tensor elements. These values are fixed by the physical details of the medium.

The suggested method may be used first of all as testing tensor calculations by other methods. Secondly, it allows one to make a first step in the problem of new artificial material synthesis because some of the physical properties and effects used in electromagnetic applications are defined by the symmetry of the medium. In the synthesis of new materials one can start from the ideal material tensors and choose those symmetries which give the desired tensor structure. The following synthesis process must be fulfilled in the framework of the chosen (anti)axis, (anti)planes and (anti)center of symmetry which form a "skeleton" of the future medium.

As a following step in the investigation of the medium properties may be application of the theory of representations and corepresentations of magnetic groups and group-theoretical perturbation method [12]. This analysis allows one to predict some general properties of the media, in particular degeneracy of plane-wave eigenvalues and removing this degeneracy by means of an external perturbation (for example, dc magnetic field). It will be discussed elsewhere.

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Appendix: Symmetry description of zero, first and second rank tensors

In this appendix, we summarize briefly some known results concerning symmetry properties of tensors.

Symmetry of a tensor is its property to be invariant under space-time transformations: proper and improper rotations, time reversal operation and combinations of space and time operations. Tensor has a symmetry element ((anti)axis, (anti)plane, (anti)center) if all the components of the tensor are transformed into themselves under the transformation corresponding to this element. Symmetry of a tensor is its intrinsic property which does not depend on the choice of coordinate system.

Symmetry properties of scalars, vectors and tensors

Scalars do not depend on direction. They may be considered as tensors of the zero rank. There are two types of scalars: real scalars (for example, an electric charge) and pseudoscalars (for example, an angle of rotation of the plane polarization in optical activity). The real scalars do not change sign under changing the sign of the coordinate system. In other words, they do not change sign under spatial reflection and inversion. The peudoscalars do change their sign under these transformations.

The real scalars possess the highest symmetry K_{h} because they are not changed under all space transformations: rotations by any angle about any axis passing through the origin, and reflections in any plane passing through the origin. The pseudoscalars on the other hand, have the symmetry K, *i.e.* they have axes but do not have planes of symmetry and the center. Notice that time may be considered as a scalar with symmetry K_{h} [11].

Vectors are the simplest directional quantities. They may be considered as the tensors of the first rank. There exist two types of the vectors: real vectors (polar ones) and pseudovectors (axial ones). The electric current $\mathbf{j}$, flux $\mathbf{D}$ and the electric field $\mathbf{E}$ are polar vectors, but the magnetic current, the magnetic flux $\mathbf{B}$ and the field $\mathbf{H}$ are axial ones [5]. The transformation law for the polar vectors is:

$$\mathbf{a}' = R\mathbf{a} \tag{A.1}$$

where $\mathbf{a}$ is a given polar vector, $\mathbf{a}'$ is a mapped vector and R is a mapping operator in 3D space. The transformation formula for the axial vectors has the following form:

$$\mathbf{A}' = (\det[R])R\mathbf{A} \tag{A.2}$$

where $\mathbf{A}$ is an axial vector, det means determinant and $[R]$ is 3D matrix representation of the operator R.

The polar vectors $\mathbf{D}$ and $\mathbf{E}$ have the symmetry $C_{\infty\mathrm{v}}$. The axial vectors $\mathbf{B}$ and $\mathbf{H}$ have the magnetic symmetry $D_{\infty\mathrm{h}}$ ($C_{\infty\mathrm{h}}$).

Tensors of the second rank are more complex directional quantities. Here, we must also distinguish between the polar and the axial tensors. The polar tensors define a linear relation between two polar vectors or between two axial ones, and the axial tensors determine a relation between an axial vector and a polar one. For example, from equations (1, 2) we see that $[\varepsilon]$ and $[\mu]$ are the polar tensors and $[\xi]$, $[\zeta]$ are the axial ones.

As in the case of the vectors, the laws of spatial transformations are different for the polar and for the axial tensors:

$$\mathbf{a}' = R\mathbf{a}R^{-1} \tag{A.3}$$

for polar tensors, and

$$\mathbf{A}' = (\det[R])R\mathbf{A}R^{-1} \tag{A.4}$$

for axial ones, *i.e.* under rotations, the axial and polar tensors transform equally, but under reflection and inversion they transform in different way. In other words, the axial tensors are sensitive to the change of the coordinate system sign.

Any tensor may be decomposed into the sum of the symmetric and antisymmetric parts. In a certain coordinate system, the symmetric part may be brought out to the diagonal form and the antisymmetric part to the simplest form

$$\begin{vmatrix} 0 & -a_{12} & 0 \\ a_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{A.5}$$

in another coordinate system (in some cases, these two coordinate systems may coincide). Notice that the symmetric and antisymmetric parts of $[\xi]$ and $[\zeta]$ describe different physical processes in the medium. The symmetric part transforms the vectors of electric and magnetic field one into another so that they lie on one straight line. It may be called a longitudinal effect. The antisymmetric part of the tensors $[\xi]$ and $[\zeta]$ transforms the electric and the magnetic vectors one into another through a polar vector and describes a transverse effect which is analogous to the Hall effect.

Space symmetry of the constitutive tensors

If a medium is described by a group of symmetry, the constitutive tensors are invariant under the operators of this group. In order to define the structure of the constitutive tensors, we may use only generators of the group [13]. Thus, the tensors may be calculated by the following relations:

$$K = PKP \tag{A.6}$$

where the 6×6 constitutive tensor K is

$$K = \begin{vmatrix} [\varepsilon] & [\xi] \\ [\zeta] & [\mu] \end{vmatrix} \tag{A.7}$$

and a 6×6 mapping operator is given by:

$$P = \begin{vmatrix} [R] & 0 \\ 0 & (\det[R])\,[R] \end{vmatrix}, \tag{A.8}$$

$\det[R]$ is included in the operator P in order to take into account the axial nature of the vector $\mathbf{H}$. The properties of P follow from the properties of R:

$$P = P^{-1} = P^t; \quad PP = I. \tag{A.9}$$

Time-reversal symmetry of the constitutive tensors

Under time reversal, electric charge, electric flux and electric field are even, but current, magnetic flux and magnetic field and Poynting vector are odd. The tensors $[\mu]$ and $[\varepsilon]$ do not change their signs under time reversal, *i.e.* they are even in time. The tensors $[\xi]$ and $[\zeta]$ on the other hand are odd in time because for example in the equation (1) the tensor $[\xi]$ must transform the odd under time reversal vector $\mathbf{H}$ into the even in time vector $\mathbf{D}$. In order to adjust these different transformation properties of $\mathbf{D}$ and $\mathbf{H}$, the tensor $[\xi]$ must be odd in time.

Notice that in the case of lossy media, the time reversal operator converts a damping wave into a growing one and *vice versa*, *i.e.* the dissipative processes are not time reversible. In order to overcome this difficulty, an artifice has been suggested in [9], namely to use the so called restricted time reversal operator. This operator does not apply to the imaginary dissipative terms of the constitutive parameters and to that of the wave vector. It preserves the damping or the growing character of the wave under time reversal.

In order to find transformation properties of the constitutive tensors under the operator T, we must turn to Onsager's theorem of irreversible thermodynamics [17]: if a force $\mathbf{F}$ induces a response $\mathbf{G}$, and they are connected by a susceptibility χ:

$$\mathbf{G} = \chi \mathbf{F}, \tag{A.10}$$

in the presence of a dc magnetic field $\mathbf{H}_0$, χ has a property

$$\chi(-\mathbf{H}_0) = \chi^t(\mathbf{H}_0). \tag{A.11}$$

In our case, the force is the six-vector $\mathbf{e}$:

$$\mathbf{e}(\mathbf{r}) = \begin{vmatrix} \mathbf{E} \\ \mathbf{H} \end{vmatrix}, \tag{A.12}$$

the response is the six-vector combined from $\mathbf{D}$ and $\mathbf{B}$, and the susceptibility is the constitutive tensor K:

$$\begin{vmatrix} \mathbf{D} \\ \mathbf{B} \end{vmatrix} = K \begin{vmatrix} \mathbf{E} \\ \mathbf{H} \end{vmatrix}. \tag{A.13}$$

The operator T acts on the time-harmonic quantities and on the operators as follows:

(1) complex conjugates all the quantities;
(2) changes the sign of $\mathbf{H}$ ($\mathbf{H}$ is odd under time inversion);
(3) transposes the operators (according to the Onsager's principle [17]).

Application of the operator T changes also the sign of an external dc magnetic field $\mathbf{H}_0$. We may write:

$$T \begin{vmatrix} \mathbf{D} \\ \mathbf{B} \end{vmatrix} = TK \begin{vmatrix} \mathbf{E} \\ \mathbf{H} \end{vmatrix} \tag{A.14}$$

or

$$\begin{vmatrix} \mathbf{D}^* \\ -\mathbf{B}^* \end{vmatrix} = \begin{vmatrix} [\varepsilon^*]^t & [\zeta^*]^t \\ [\xi^*]^t & [\mu^*]^t \end{vmatrix} \begin{vmatrix} \mathbf{E}^* \\ -\mathbf{H}^* \end{vmatrix}. \tag{A.15}$$

Complex conjugation of (A.15) gives:

$$\mathbf{D} = [\varepsilon]^t \mathbf{E} - [\zeta]^t \mathbf{H}, \tag{A.16}$$

$$\mathbf{B} = -[\xi]^t \mathbf{E} + [\mu]^t \mathbf{H}. \tag{A.17}$$

These two equations correspond to the complementary [2], or Lorentz-adjoint medium [9]. Comparing equations (A.16, A.17) with equation (A.7) we see that invariantness of the media under time reversal corresponds to the reciprocity conditions (3).

Combined space-time symmetry of the constitutive tensors

If a medium is described by the magnetic groups of the third category, in order to obtain the constitutive tensors we must use along with unitary operators an antiunitary one as well [21]. An antiunitary operator is a combined operator TP. Application of this operator to the constitutive relations gives:

$$K = PK^tP. \tag{A.18}$$

We can summarize the above results as follows. From invariantness of a medium under space transformation (A.6) we obtain the expressions:

$$[R][\varepsilon] = [\varepsilon][R],\ [R][\mu] = [\mu][R] \tag{A.19}$$

$$[R][\xi] = \det([R])[\xi][R],\ [R][\zeta] = \det([R])[\zeta][R] \tag{A.20}$$

and under space-time transformations (A.18):

$$[R][\varepsilon] = [\varepsilon]^t[R], [R][\mu] = [\mu]^t[R], \tag{A.21}$$

$$[R][\xi] = -\det([R])[\zeta]^t[R],\ [R][\zeta] = -\det([R])[\xi]^t[R]. \tag{A.22}$$

The relations (A.19–A.22) have been obtained in [8] from the symmetry consideration of Maxwell's equations.

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