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High-gradient electron acceleration in a plasma-loaded wiggler

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Abstract. Detailed derivations and further analysis are presented of a recent concept for a plasma-based accelerator scheme incorporating a strong circularly polarised magnetic wiggler field producing relativistic-strength diamagnetic transverse plasma currents. The increase in the plasma Lorentz factor leads to a substantial increase in the longitudinal component of the wave electric field and therefore of the acceleration rate. It is also found that ultra-high acceleration gradients are possible with relatively low plasma densities and long wave lengths. It also appears possible that the transverse wiggling motion of the electrons of the beam is able to delay the dephasing with the accelerating wave leading to much higher values of the energy gained by the beam at saturation and even electron bunches that have been injected with the "wrong" phase seem to be able to reverse their motion and accelerate to very high energies in short distances.

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1 Introduction

The strong longitudinal electric field of a Langmuir wave excited in a cold plasma can be used to produce high gradient acceleration of electron beams. This was predicted by Tajima and Dawson in 1979 [1] and the potentialities of this method have by now been amply demonstrated in a number of experiments [2–10].

When the ion dynamics can be ignored, the longitudinal component of the electric field of a plasma wave scales as

$$E_{\parallel} = 4\pi e\delta n/k \approx 10^2 (v_{\mathrm{f}}/c)(\delta n/n_{\mathrm{p}})\sqrt{n_{\mathrm{p}}(\mathrm{cm}^{-3})}(\omega_{\mathrm{p}}/\omega) \quad (\mathrm{V/m}) \quad (1)$$

where ω and $v_{\mathrm{f}} = \omega/k$ are the frequency and phase velocity of the wave, n_{p} the ambient plasma density, $\omega_{\mathrm{p}} = (4\pi e^2 n_{\mathrm{p}}/m)^{1/2}$ the electron plasma frequency and $\delta n/n_{\mathrm{p}}$ the relative density perturbation associated with the wave.

In a plasma which is not submitted to external magnetic fields the accelerating wave is purely longitudinal and has a frequency $\omega \approx \omega_{\mathrm{p}}$. The necessity of operating, in these cases, with high-density plasmas with n_{p} greater than, say, 10^{16} cm^{-3}, follows immediately from the preceding estimate of the electric field. The acceleration process in a nonmagnetic plasma is due to the action of large amplitude Langmuir waves with phase velocity close to but less than c, the velocity of light in vacuum and saturates rather rapidly when the particle velocities exceed the phase velocity of the wave by a definite amount, the electrons of the beam gaining only a finite energy [11].

By superposing a uniform magnetic field perpendicular to the beam axis in the "surfatron" scheme [12,13] one is able to overcome the limitation due to the rapid dephasing of the electrons. In fact, the electrons that are trapped by the wave in a surfatron move along the front of the wave gaining energy in an unlimited way. The process, however, leads to several difficulties: first, the acceleration may become diffusive when the magnetic field is strong enough to detrap the electrons out of the potential wells of the wave [14–16]. Furthermore, the accelerating field of the surfatron is weaker than that of the plasma beat-wave or wake-field accelerators, because the wave-particle interaction takes place at $v_{\mathrm{f}} \ll c$ and the frequency ω of the wave is larger than ω_{p}. Finally, the transverse component of the electron motion may well result in deterioration of the beam quality, not to mention requiring a much wider or oddly angled plasma to follow the accelerated electrons.

Our group has recently proposed an accelerating scheme that has the advantage of the surfatron, namely a good phasing between the electrons and the wave and is free from its inconvenient features [17,18]. The idea is basically that of placing the plasma in the spatially periodic magnetic field of a free electron laser "wiggler". We shall refer to this scheme as to the wiggler plasma wave accelerator (WPWA).

Quite apart from achieving the possible surfatron benefits of reducing the phase mismatch and reducing or eliminating the lateral beam "walk-off" of the surfatron, there can be dramatic gains in acceleration field as well.

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The acceleration electric field for a given density modulation (Eq. (1)), at near-light phase velocity can have an improvement proportional to the acceleration wave's wavelength, which is to say, proportional to ω_p/ω. If the plasma is starting from an equilibrium with zero-current, there is little change in this parameter and so little improvement. However, in the case in which the plasma is in a diamagnetic equilibrium, carrying strong transverse currents, for external wiggler fields so high that the plasma currents are electron-relativistic, the effective plasma frequency can be dramatically reduced by the associated increase in the electron plasma Lorentz factor γ_p with concomitant increase in accelerator field for a given $\delta n/n_p$. This aspect was initially explored in earlier work [18] and is given more detailed treatment here.

Leaving explicitly aside important practical questions of how to reach or maintain this diamagnetic equilibrium, in this work, after a careful and detailed review of the derivation of the essential equations, the accelerator-related features of the diamagnetic case are explored further by calculating acceleration behaviour of injected electrons for a given density modulation and for various choices for the other parameters.

As we have said, the wiggler field may be able to induce inside the plasma diamagnetic currents that tend to screen it from the external magnetic field. For high values of the external field these currents become relativistic and serve to modify in a favourable way the dispersion properties of the plasma. In fact, if the plasma is carrying very high currents we may define a plasma Lorentz factor γ_p in the usual way and show that, as a result of the relativistic increase of the electron mass, the frequencies ω of all waves inside the plasma decrease with the increase of γ_p according to the inequality

$$\gamma_p^{-3/2} < \omega/\omega_p < \gamma_p^{-1/2}.$$

The Lorentz factor turns out to be approximately given by the expression $\gamma_p \approx a_{w0} - \omega_p^2/c^2k_w^2$ for large values of the "wiggler parameter" a_{w0}. This last parameter is defined by the relation $a_{w0} = eB_w/mc^2k_w = \omega_c/ck_w$, where B_w and $\lambda_w = 2\pi/k_w$ are peak-value and spatial periodicity, respectively, of the wiggler field, while ω_c is the cyclotron frequency of the electron in the magnetic field of the wiggler. It follows that, when $a_{w0} \gg 1$, γ_p is also a large number and the wave frequency becomes smaller than the plasma frequency ω_p. We have then, from (1), that the accelerating electric field of the wave becomes larger than the field of the Langmuir wave excited in nonmagnetic plasmas by the extra factor ω_p/ω which has values from $\gamma_p^{1/2}$ up to $\gamma_p^{3/2}$. The increase in the accelerating component of the wave electric field eventually leads to larger values of the rate of acceleration of the electron beam in MeV per meter.

In addition, the wiggler field with its definite periodicity in space induces a characteristically wiggling transverse motion of the beam that may allow the electrons to maintain their phases relative to the wave for longer time intervals, as it happens in the surfatron scheme, so that the saturation of the acceleration process, which is due to the dephasing of the beam and/or to the depletion of the pump, takes place over much longer time (or space) intervals, the wiggling motion being such that the electrons of the beam do not practically move away from the beam axis. All details referring to the transverse motion of the electrons, should, however, find their proper place in a theory that takes into account the complete, *i.e.*, off-axis structure of the wiggler field, as well as the transverse profiles of the electron bunch and of the wave pulses.

In Section 2 a full derivation and discussion is presented of the equations used in the previous work [17,18], together with some conclusions drawn from the dispersion relation for the wiggler plasma waves. In Section 3, after recalling a re-presentation of an acceleration result from Figure 2 from [18], a discussion is given concerning an estimate for the accelerating field to be obtained from the dispersion relation, following which new numerical results from detailed acceleration calculations in various conditions are presented and discussed for finite-length injected electron bunches. The conclusions are given in Section 4.

2 1D equations

We consider a cold plasma placed inside the wiggler cavity of a free electron laser. The wiggler magnetic field is helical with the spatial periodicity λ_w and is given by $\mathbf{B}_w(z) = \mathbf{e}_z \times \frac{\partial}{\partial z}\mathbf{A}_w(z)$ where the transverse vector potential $\mathbf{A}_w(z)$ is

$$\mathbf{A}_w(z) = \frac{a_w}{\sqrt{2}}(\hat{\mathbf{e}}\mathrm{e}^{-\mathrm{i}k_w z} + cc) \tag{2}$$

$\hat{\mathbf{e}} = (1/\sqrt{2})(\mathbf{e}_x + \mathrm{i}\mathbf{e}_y)$ and $k_w = 2\pi/\lambda_w$.

The electrons of the plasma move with the (relativistic) fluid velocity $\mathbf{u}(z,t)$, the corresponding momentum being $\mathbf{p}(z,t) = m\gamma_p(z,t)\mathbf{u}(z,t)$ ($\gamma_p = (1-\mathbf{u}^2/c^2)^{-1/2}$), while the positive ions are immobile and distributed with the number density n_p. The basic fluid equations describing the electron dynamics express, as usual, the conservation of the electron number and momentum. If the electromagnetic radiation fields $\mathbf{E}$ and $\mathbf{B}$ are written in terms of a transverse vector potential $\mathbf{A}(z,t)$, namely if $\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} + \mathbf{e}_z E_z$ and $\mathbf{B} = \mathbf{e}_z \times \frac{\partial}{\partial z}\mathbf{A}$, these equations have the following form

$$\frac{\partial}{\partial t}n + \frac{\partial}{\partial z}(nu_z) = 0 \tag{3}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{p} + u_z\frac{\partial}{\partial z}\mathbf{p} &= -e\mathbf{E} - \frac{e}{c}\mathbf{u}\times(\mathbf{B}+\mathbf{B}_w)\\ &= \frac{e}{c}\frac{\partial}{\partial t}\mathbf{A} + \frac{e}{c}u_z\frac{\partial}{\partial z}(\mathbf{A}+\mathbf{A}_w)\\ &\quad -\mathbf{e}_z\left[eE_z + \frac{e}{c}\mathbf{u}_\perp\cdot\frac{\partial}{\partial z}(\mathbf{A}+\mathbf{A}_w)\right]\cdot\end{aligned} \tag{4}$$

$\mathbf{u}_\perp$ is the transverse component of the electron fluid velocity.

By projecting the momentum equation (4) upon the (x,y) plane, one obtains the conservation equation for the generalised perpendicular momentum density $\mathbf{Q} = n(\mathbf{p}_\perp - \frac{e}{c}(\mathbf{A}+\mathbf{A}_\mathrm{w}))$, *i.e.*

$$\frac{\partial}{\partial t}\mathbf{Q} + \frac{\partial}{\partial z}(u_z\mathbf{Q}) = 0. \tag{5}$$

If we assume that $\mathbf{Q}(z,t)$ is identically zero at some time $t = 0$, it follows that

$$\mathbf{p}_\perp = \frac{e}{c}(\mathbf{A}+\mathbf{A}_\mathrm{w}) \tag{6}$$

for all times. While this is the normal case for the variation from equilibrium, whether it should apply to the equilibrium itself depends on how that equilibrium was obtained.

The longitudinal dynamics is described by:

$$\frac{\partial}{\partial t}p_z + u_z\frac{\partial}{\partial z}p_z = -eE_z - \frac{e^2}{2mc^2\gamma_\mathrm{p}}\frac{\partial}{\partial z}(\mathbf{A}+\mathbf{A}_\mathrm{w})^2. \tag{7}$$

The axial component $E_z(z,t)$ of the electric field and the radiation potential $\mathbf{A}(z,t)$ satisfy the Maxwell equations

$$\frac{\partial}{\partial z}E_z = 4\pi(\rho_\mathrm{b}+\rho_\mathrm{p}) \tag{8}$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2}\right)\mathbf{A} = 4\pi c(\mathbf{J}_\mathrm{b}+\mathbf{J}_\mathrm{p}) \tag{9}$$

where both beam and plasma contributions are present. In fact, the quantities

$$\rho_\mathrm{b} = -en_\mathrm{s}\sum_j \delta(z - z_j(t)) \tag{10}$$

$$\mathbf{J}_\mathrm{b} = -en_\mathrm{s}\sum_j \mathbf{v}_j(t)\delta(z - z_j(t)) \tag{11}$$

are the beam charge and current densities defined according to the "charged plane model" as used, *e.g.*, in [19], where n_s is the surface charge density on each plane. Furthermore

$$\rho_\mathrm{p} = e(n_\mathrm{p} - n) \tag{12}$$

gives the deviation from neutrality in the plasma, while the transverse component of the plasma current density $\mathbf{J}_\mathrm{p} = -en\mathbf{u}$ can be written as

$$\mathbf{J}_{p_\perp} = -en\mathbf{u}_\perp = -\frac{e^2 n}{mc\gamma_\mathrm{p}}(\mathbf{A}+\mathbf{A}_\mathrm{w}) \tag{13}$$

now that the assumption (*i.e.* $\mathbf{Q} = 0$ at $t = 0$) *is* made so that equation (6) can be used in equation (13).

Finally, the electrons of the beam satisfy the relativistic equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}z_j(t) = v_{j\parallel}(t) \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_j(t) = -\{eE_z(z,t) + \frac{e^2}{2mc^2\gamma_j(t)}\frac{\partial}{\partial z}(\mathbf{A}+\mathbf{A}_\mathrm{w})^2\}_{z=z_j(t)} \tag{15}$$

where $\pi_j(t) = m\gamma_j(t)v_{j\parallel}(t)$ is the parallel momentum, $\gamma_j(t)$ the Lorentz factor and we have used the equation

$$\mathbf{v}_{j\perp} = \frac{e}{mc\gamma_j(t)}(\mathbf{A}+\mathbf{A}_\mathrm{w})_{z=z_j(t)} \tag{16}$$

which establishes a direct relationship between the transverse motion of the electrons of the beam and the total vector potential.

2.1 Equilibrium of the plasma in the absence of the beam

We assume that the plasma is in equilibrium with the static magnetic field of the wiggler before the injection of the electron beam. As usual, there are a number of possible stationary solutions to the basic equations (3, 7, 8, 9). We envisage a stationary situation in which the plasma is uniform in space, *i.e.*, $n_\mathrm{EQ} = n_\mathrm{p} = const$, and has no axial currents and no axial electric fields, namely $u_z = 0$ and $E_z = 0$. By putting $\mathbf{J}_\mathrm{b} = 0$ and dropping the time derivative, equation (9) gives the potential vector $\mathbf{A}_\mathrm{EQ}(z)$ in equilibrium

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\mathbf{A}_\mathrm{EQ}(z) - \frac{\omega_\mathrm{p}^2}{c^2\gamma_\mathrm{p0}}\mathbf{A}_\mathrm{EQ}(z) = \frac{\omega_\mathrm{p}^2}{c^2\gamma_\mathrm{p0}}\mathbf{A}_\mathrm{w}(z)$$

where $\gamma_\mathrm{p0} = const$ is the equilibrium value of the plasma Lorentz factor associated with the equilibrium vector potential value $\mathbf{A}_\mathrm{EQ}$. From this equation one has

$$\mathbf{A}_\mathrm{EQ}(z) = -\frac{\dfrac{\omega_\mathrm{p}^2}{c^2k_\mathrm{w}^2}}{\gamma_\mathrm{p0} + \dfrac{\omega_\mathrm{p}^2}{c^2k_\mathrm{w}^2}}\mathbf{A}_\mathrm{w}(z) \tag{17}$$

and therefore the total potential vector within the plasma is

$$\mathbf{A}_\mathrm{TOT} = \frac{\gamma_\mathrm{p0}}{\gamma_\mathrm{p0} + \dfrac{\omega_\mathrm{p}^2}{c^2k_\mathrm{w}^2}}\mathbf{A}_\mathrm{w}. \tag{18}$$

One can see that, as a consequence of equation (6), the plasma is carrying (diamagnetic) currents that flow along the wiggler field lines with the density

$$\mathbf{J}_\mathrm{EQ} = -en_\mathrm{p}\mathbf{u}_{\perp\mathrm{EQ}} = -\frac{1}{4\pi c}\frac{\omega_\mathrm{p}^2}{\gamma_\mathrm{p0} + \dfrac{\omega_\mathrm{p}^2}{c^2k_\mathrm{w}^2}}\mathbf{A}_\mathrm{w}\,. \tag{19}$$

One then defines the factor $\beta_\mathrm{p0} = |\mathbf{u}_{\perp\mathrm{EQ}}|/c$ which, according to the preceding equation and the definition of the wiggler parameter $a_\mathrm{w0} = \frac{ea_\mathrm{w}}{mc^2}$ can be written as

$$\beta_\mathrm{p0} = \frac{a_\mathrm{w0}}{\gamma_\mathrm{p0} + \dfrac{\omega_\mathrm{p}^2}{c^2k_\mathrm{w}^2}}\,. \tag{20}$$

By recalling that $\beta_\mathrm{p0} = (1 - \gamma_\mathrm{p0}^{-2})^{1/2}$, one sees that one now has an implicit equation for γ_p0 in terms of the two parameters $\omega_\mathrm{p}/ck_\mathrm{w}$ and a_w0.

2.2 Passage of the electron beam inside the plasma

We assume that an accelerating packet of rather strong (*i.e.* such that $\delta n/n_\mathrm{p}$ is not necessarily very small) but still linear waves has been excited inside the plasma. A beam of electrons with a volume density n_b such that

$$n_\mathrm{b} \ll n_\mathrm{p} \tag{21}$$

is then injected into the plasma. On account of this inequality and the linear character of the accelerating wave packet, we may write all field quantities in the form $n = n_\mathrm{p} + \delta n$, $u_z = \delta u_z$, $\gamma_\mathrm{p} = \gamma_{\mathrm{p}0} + \delta\gamma$, $E_z = \delta E_z$ and $\mathbf{A} = \mathbf{A}_\mathrm{EQ} + \delta\mathbf{A}$, where δn, $\delta\mathbf{A}$, etc. are small disturbances. It is also convenient to express the perturbation $\delta\mathbf{A}$ in the frame of the helical wiggler, *i.e.*, to write

$$\delta\mathbf{A} = \delta A\hat{\mathbf{e}} + cc \tag{22}$$

and then change to the non dimensional quantity $A = e\delta A/mc^2$.

From equations (3, 7, 8, 9), one can deduce the following equations for the two numbers A and $\delta n/n_\mathrm{p}$:

$$\begin{aligned}\left[\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2} + \frac{\omega_\mathrm{p}^2}{\gamma_{\mathrm{p}0}}(1-\frac{1}{2}\beta_{\mathrm{p}0}^2)\right] A \\ - \frac{\omega_\mathrm{p}^2\beta_{\mathrm{p}0}^2}{2\gamma_{\mathrm{p}0}}\mathrm{e}^{-2\mathrm{i}k_\mathrm{w}z}A^* + \frac{\omega_\mathrm{p}^2\beta_{\mathrm{p}0}}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}k_\mathrm{w}z}\frac{\delta n}{n_\mathrm{p}} = \\ -\frac{4\pi e^2 n_\mathrm{s}\gamma_{\mathrm{p}0}\beta_{\mathrm{p}0}}{\sqrt{2}\,m}\sum_j \frac{1}{\gamma_j}\mathrm{e}^{-\mathrm{i}k_\mathrm{w}z}\delta(z - z_j(t))\end{aligned} \tag{23}$$

$$\begin{aligned}\left[\frac{\partial^2}{\partial t^2} + \frac{\omega_\mathrm{p}^2}{\gamma_{\mathrm{p}0}}\right]\frac{\delta n}{n_\mathrm{p}} - \frac{c^2\beta_{\mathrm{p}0}}{\sqrt{2}\,\gamma_{\mathrm{p}0}}\frac{\partial^2}{\partial z^2}(\mathrm{e}^{\mathrm{i}k_\mathrm{w}z}A + cc) = \\ -\frac{4\pi e^2 n_\mathrm{s}}{m\gamma_{\mathrm{p}0}}\sum_j \delta(z - z_j(t)).\end{aligned} \tag{24}$$

There is also the adjoint equation for equation (23) which, unlike equation (24) is not self-adjoint, and so is needed to include A^* correctly.

On account of the weakness of the beam as specified in (21), the rhs of these two equations are small quantities of the order of $n_\mathrm{b}/n_\mathrm{p} \ll 1$. In addition, in writing these equations all non linear terms of second or higher order in the parameters of smallness that represents the weakness of the disturbance have been disregarded. Equations (23, 24) still present the disadvantage of having coefficients that depend on the space variable z. A system with constant coefficients, however, can be readily obtained by making the simple change of variable from A to $\tilde{A} = A\exp(\mathrm{i}k_\mathrm{w}z)$. While changing the A variable, one should mention that, because the equilibrium electron density is uniform (unlike the equilibrium vector potential $\mathbf{A}_\mathrm{w}(z)$), one does not have to employ a similar change of variables for $\delta n/n_\mathrm{p}$.

The two equations then read

$$\begin{aligned}\left[\frac{\partial^2}{\partial t^2} - c^2\left(\frac{\partial}{\partial z} - \mathrm{i}k_\mathrm{w}\right)^2 + \frac{\omega_\mathrm{p}^2}{\gamma_{\mathrm{p}0}}(1-\frac{1}{2}\beta_{\mathrm{p}0}^2)\right]\tilde{A} \\ - \frac{\omega_\mathrm{p}^2\beta_{\mathrm{p}0}^2}{2\gamma_{\mathrm{p}0}}\tilde{A}^* + \frac{\omega_\mathrm{p}^2\beta_{\mathrm{p}0}}{\sqrt{2}}\frac{\delta n}{n_\mathrm{p}} = \\ -\frac{4\pi e^2 n_\mathrm{s}\gamma_{\mathrm{p}0}\beta_{\mathrm{p}0}}{\sqrt{2}\,m}\sum_j \frac{1}{\gamma_j}\delta(z - z_j(t)).\end{aligned} \tag{25}$$

$$\begin{aligned}\left[\frac{\partial^2}{\partial t^2} + \frac{\omega_\mathrm{p}^2}{\gamma_{\mathrm{p}0}}\right]\frac{\delta n}{n_\mathrm{p}} - \frac{c^2\beta_{\mathrm{p}0}}{\sqrt{2}\,\gamma_{\mathrm{p}0}}\frac{\partial^2}{\partial z^2}(\tilde{A} + cc) \\ = -\frac{4\pi e^2 n_\mathrm{s}}{m\gamma_{\mathrm{p}0}}\sum_j \delta(z - z_j(t)).\end{aligned} \tag{26}$$

In the same way as for equation (23), there is the adjoint equation for equation (25), where $\tilde{A}$ and $\tilde{A}^*$ exchange roles.

At last, the description is closed by adding the two equations (14, 15) that give the axial dynamics of the beam and that can be consistently written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}z_j(t) = v_j(t) = c\beta_j(t) \tag{27}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}p_j(t) &= -\left\{\frac{e}{mc}\delta E_z(z,t) + \frac{e^2}{m^2c^3\gamma_j(t)}\frac{\partial}{\partial z}(\mathbf{A}_\mathrm{TOT}\cdot\delta\mathbf{A})\right\}_{z=z_j(t)} \\ &= -\left\{\frac{e}{mc}\delta E_z(z,t) + \frac{\beta_{\mathrm{p}0}\gamma_{\mathrm{p}0}c}{\sqrt{2}\gamma_j(t)}\frac{\partial}{\partial z}(A\mathrm{e}^{\mathrm{i}k_\mathrm{w}z} + cc)\right\}_{z=z_j(t)} \\ &= -\left\{\frac{e}{mc}\delta E_z(z,t) + \frac{\beta_{\mathrm{p}0}\gamma_{\mathrm{p}0}c}{\sqrt{2}\gamma_j(t)}\frac{\partial}{\partial z}(\tilde{A} + cc)\right\}_{z=z_j(t)}\end{aligned} \tag{28}$$

with $\mathbf{A}_\mathrm{TOT}$ defined in (18) and $p_j(t) = \beta_j(t)\gamma_j(t)$ is the parallel momentum normalised to mc.

2.3 Plasma-wiggler dispersion relation

The dispersion relation of the plasma in the presence of the strong field of the wiggler but without the electron beam, can be obtained by equating to zero the right hand sides of the two equations (25, 26) and separating equation (25) into the two equations for the real and imaginary parts $\tilde{A}_\mathrm{r}$ and $\tilde{A}_\mathrm{i}$ of the complex quantity $\tilde{A}$. The resulting system is then solved in terms of (normal) solutions of the type $\tilde{A}_\mathrm{r} = \tilde{A}_{\mathrm{r}0}\exp(\mathrm{i}kz - \mathrm{i}\omega t)$, $\tilde{A}_\mathrm{i} = \tilde{A}_{\mathrm{i}0}\exp(\mathrm{i}kz - \mathrm{i}\omega t)$ and $\delta n/n_\mathrm{p} = (\delta n/n_\mathrm{p})_0\ \exp(\mathrm{i}kz - \mathrm{i}\omega t)$. We get in the end a linear and homogeneous system of equations for the amplitudes $\tilde{A}_{\mathrm{r}0}$, $\tilde{A}_{\mathrm{i}0}$, and $(\delta n/n_\mathrm{p})_0$ which is given next:

$$\begin{aligned}\Big[-\omega^2 + c^2(k^2 + k_\mathrm{w}^2) + \frac{\omega_\mathrm{p}^2}{\gamma_{\mathrm{p}0}}(1 - \beta_{\mathrm{p}0}^2)\Big]\tilde{A}_{\mathrm{r}0} \\ + \frac{\omega_\mathrm{p}^2\beta_{\mathrm{p}0}}{\sqrt{2}}(\delta n/n_\mathrm{p})_0 \ = \ 0\end{aligned}$$

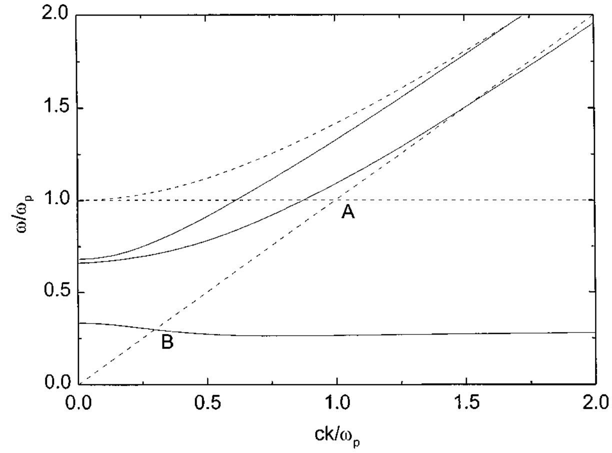

Fig. 1. The three branches of the dispersion relation (29) (solid lines) for $\beta_{p0} = 0.9$ and $\omega_p/ck_w = 6$. The figure also gives the two branches of the non magnetic case (dashed lines). Points A and B are the intersections of the two longitudinal branches with the straight line $\omega = ck$ and indicate the waves that must be excited in the non magnetised and magnetised plasma, respectively.

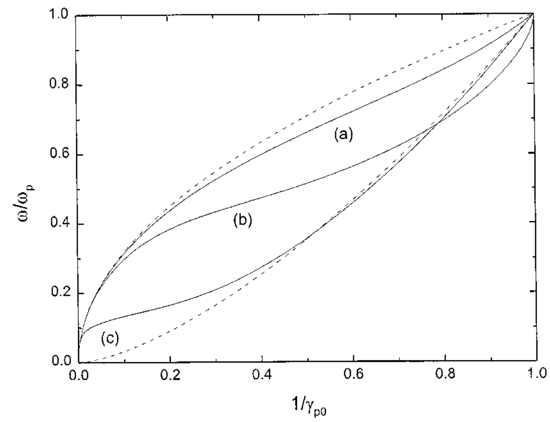

Fig. 2. The ratio ω/ω_p *vs.* $1/\gamma_{p0}$ for the lower quasi-longitudinal branch of Figure 1 and for waves whose phase velocity is equal to c, *i.e.*, $\omega/ck = 1$. The solid curves (a), (b) and (c) are for $\omega_p/ck_w = 0.5$, 1 and 5. The upper and lower dashed lines give $(1/\gamma_{p0})^{1/2}$ and $(1/\gamma_{p0})^{3/2}$.

$$2ic^2 k_w k\tilde{A}_{r0} + \Big[-\omega^2 + c^2(k^2+k_w^2) + \frac{\omega_p^2}{\gamma_{p0}}\Big]\tilde{A}_{i0} = 0$$

$$2\frac{c^2k^2\beta_{p0}}{\sqrt{2}\gamma_{p0}} + \Big[-\omega^2 + \frac{\omega_p^2}{\gamma_{p0}}\Big](\delta n/n_p)_0 = 0.$$

By equating to zero the determinant of the matrix of this set of equations, we get the dispersion relation given next in the form of equation (1) of [18]

$$\theta^3 - 2\theta^2[c^2(k^2+k_w^2) - \Gamma] + \theta[c^4(k^2-k^2{}_w)^2 - 4\Gamma c^2k^2 - 2\Gamma c^2k_w^2] + 2\Gamma c^4k^2(k^2+k_w^2) = 0 \quad (29)$$

where $\theta = \omega^2 - \omega_p^2/\gamma_{p0}, \Gamma = \omega_p^2\beta_{p0}^2/2\gamma_{p0}$. We can also write the polarisation ratio

$$\tilde{A}_0 = \frac{\gamma_{p0}\theta(\theta - c^2(k+k_w)^2)}{\sqrt{2}\beta_{p0}c^2k^2(\theta - c^2(k^2+k_w^2))}(\delta n/n_p)_0. \quad (30)$$

The algebraic equation (29) is of third degree in ω^2, which means that there are three branches in the dispersion relation. These three eigen-frequencies are represented in Figure 1 for a particular choice of the two basic parameters β_{p0} and ω_p/ck_w. For comparison, this figure gives also the two branches of the nonmagnetic case.

In the presence of the wiggler magnetic field all frequencies are lowered as a result of the strong currents induced in the plasma and the corresponding relativistic increase of the electron mass. In particular, the lower branch in Figure 1 (which is Fig. 1 from [18]) develops entirely in a region where $\omega(k) < \omega_p$. This is the branch that goes back to the Langmuir branch in the limit in which the wiggler field tends to zero and has always a larger longitudinal component of the electric field. The other two branches are basically electromagnetic and come from the splitting and shifting on the k-axis of the original transverse branch, due to the large wiggler magnetic field. The accelerating wave-packet that is excited in the plasma must, in any case, have a carrier whose phase velocity is as near as possible to the velocity of light c. This wave is represented in Figure 1 by point B, the intersection of the lower quasi-longitudinal branch of the dispersion relation with the straight line $\omega = ck$. The point A in the same figure gives instead the Langmuir wave which is excited in the non magnetic case. According to the estimate in (1), the fact that the accelerating wave has a frequency lower than ω_p must result in a corresponding increase in the longitudinal field of the wave and eventually in a higher value of the acceleration rate. A more complete idea of the increase in the electric field is given in Figure 2, which shows the factor ω/ω_p for the lower branch of the dispersion relation. This quantity depends only on the two parameters ω_p/ck_w and γ_{p0} and in Figure 2 we give it as a function of $1/\gamma_{p0}$ for various fixed values of the other parameter. One can see that ω/ω_p has very small values only for large values of γ_{p0}, larger, in any case, or even much larger than those of the other parameter ω_p/ck_w. This means that the plasma must be strongly magnetised, *i.e.*, the cyclotron frequency ω_c of the electrons of the plasma in the magnetic field of the wiggler must be greater or even much greater than ω_p.

2.4 Solution of the basic set of equations

Equations (25–28) may be solved through the use of the slowly varying envelope approximation, the reason for using a perturbation procedure lying in the right hand sides of the two equations (25, 26) being small quantities of the order of n_b/n_p. As a result of the perturbation method used, to zero-order in the treatment, the relative density

$\delta n/n_\mathrm{p}$ has the form

$$\frac{\delta n}{n_\mathrm{p}} = M_\mathrm{L}(z,t)\mathrm{e}^{\mathrm{i}(kz-\omega(k)t)} + cc \tag{31}$$

where the complex amplitude $M_\mathrm{L}(z,t)$ is a slowly varying function of both z and t that satisfies the equation

$$\left(\frac{\partial}{\partial t} + \omega'(k)\frac{\partial}{\partial z}\right) M_\mathrm{L}(z,t) = -\mathrm{i}C_1 \sum_{j=1}^{N} \frac{1}{\gamma_j(t)} \mathrm{e}^{-\mathrm{i}\theta_j(t)}\delta(z-z_j(t)) - \mathrm{i}C_2 \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\theta_j(t)}\delta(z-z_j(t)). \tag{32}$$

Here $\omega'(k)$ is the group velocity of the accelerating wave-packet, N the total number of the electrons of the beam, $\theta_j(t) = kz_j(t) - \omega(k)t$ the phase of the electron in the field of the wave and the two constants C_1 and C_2 have the following explicit but rather complex expressions

$$C_1 = \left(\frac{4\pi e^2 n_\mathrm{s}}{m}\right) \frac{\gamma_\mathrm{p}\beta_\mathrm{p}^2 c^2 k^2}{2\omega Z(k)} \times \left(\omega^2 - \frac{\omega_\mathrm{p}^2}{\gamma_\mathrm{p}} - c^2(k^2+k_\mathrm{w}^2)\right)\left(\omega^2 - \frac{\omega_\mathrm{p}^2}{\gamma_\mathrm{p}}\right)$$

$$C_2 = \left(\frac{4\pi e^2 n_\mathrm{s}}{m}\right) \frac{\omega_\mathrm{p}^2\beta_\mathrm{p}^2 c^2 k^2}{2\omega\gamma_\mathrm{p} Z(k)} \left(\omega^2 - \frac{\omega_\mathrm{p}^2}{\gamma_\mathrm{p}} - c^2(k^2+k_\mathrm{w}^2)\right)$$

where

$$Z(k) = 2\gamma_\mathrm{p}\left(\omega^2 - \frac{\omega_\mathrm{p}^2}{\gamma_\mathrm{p}}\right)^2 \left(\omega^2 - \frac{\omega_\mathrm{p}^2}{\gamma_\mathrm{p}} - c^2(k^2+k_\mathrm{w}^2)\right) + \omega_\mathrm{p}^2\beta_\mathrm{p}^2\left(\left(\omega^2 - \frac{\omega_\mathrm{p}^2}{\gamma_\mathrm{p}}\right)^2 - c^4k^2(k^2+k_\mathrm{w}^2)\right).$$

Equation (32) requires the corresponding slow time-scale dynamics of the electrons of the beam which comes from equations (27, 28). Once applied to these last two equations, the same perturbation procedure leads to the final form

$$\frac{\mathrm{d}}{\mathrm{d}t}z_j(t) = c\beta_j(t) = c\frac{p_j}{\sqrt{p_j^2+\gamma_{\mathrm{p}0}^2}} \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}p_j(t) = -\mathrm{i}\frac{\omega_\mathrm{b}^2}{2ck}(b\mathrm{e}^{\mathrm{i}\theta_j(t)} - cc) - \mathrm{i}\left[\frac{\omega_\mathrm{p}^2}{ck} + \frac{\gamma_{\mathrm{p}0}^2}{ck\gamma_j(t)}(\omega^2(k) - \frac{\omega_\mathrm{p}^2}{\gamma_{\mathrm{p}0}})\right] \times (M_\mathrm{L}(z_j(t),t)\mathrm{e}^{\mathrm{i}\theta_j(t)} - cc) \tag{34}$$

where $\gamma_j(t) = \sqrt{p_j^2+\gamma_{\mathrm{p}0}^2}$ and $b = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{-\mathrm{i}\theta_j}(t)$ is the bunching factor of the electron beam.

A useful estimate of the acceleration gradient can be obtained from equation (34). In fact, if we take this equation when the acceleration process has already appreciably increased the average energy of the beam and simply drop the first term in its rhs that takes into account the Coulomb interactions between the electrons of the beam and also the term inversely proportional to $\gamma_j(t)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}p_j(t) \approx -\mathrm{i}\frac{\omega_\mathrm{p}^2}{ck}(M_\mathrm{L}(t)\mathrm{e}^{\mathrm{i}\theta_j(t)} - cc)$$

or, averaging over all the electrons of the bunch

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p\rangle \approx 2\frac{\omega_\mathrm{p}^2}{ck}|M_\mathrm{L}||b|$$

where $b = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{-\mathrm{i}\theta_j}$ is the bunching factor of the whole beam and $\langle p\rangle = \frac{1}{N}\sum_{j=1}^{N}p_j$. Let us now define $w = mc^2\langle\gamma\rangle = mc^2\frac{1}{N}\sum_{j=1}^{N}\gamma_j(t)$ as the average energy of the beam and the rate of acceleration G as

$$G = \frac{\mathrm{d}w}{\mathrm{d}z}\cdot \tag{35}$$

If we assume that $\langle p\rangle \approx \langle\gamma\rangle$ and that $c\mathrm{d}t \approx \mathrm{d}z$ we may write the following equation for G directly in MeV per meter

$$G \approx 10^2\frac{\omega_\mathrm{p}^2}{ck}k(\mathrm{cm}^{-1})|M_\mathrm{L}||b| \qquad (\mathrm{MeV/m}).$$

With the aid of equation (1) and remembering that $|M_\mathrm{L}| = (1/2)(\delta n/n_\mathrm{p})_\mathrm{peak}$, we may also write

$$G \approx 9.4\times10^{-7}E_\parallel(\mathrm{Volt/m})|b| \qquad (\mathrm{MeV/m}) \tag{36}$$

where $E_\parallel$ is the accelerating electric field of the wave in Volt per meter, or even

$$G \approx g(\delta n/n_\mathrm{p})_\mathrm{peak}|b| \qquad (\mathrm{MeV/m}) \tag{37}$$

where the factor g is given explicitly by

$$g = 10^{-4}\left(\frac{\omega_\mathrm{p}}{\omega}\right)\left(\frac{\omega}{ck}\right)\sqrt{n_\mathrm{p}(\mathrm{cm}^{-3})} \qquad (\mathrm{MeV/m}). \tag{38}$$

Equation (36) says that the acceleration rate is proportional to the longitudinal component $E_\parallel$ of the electric field of the wave, as was expected. Equation (37) shows that G is really the product of three factors and if g is the factor given in equation (38), the other two factors $(\delta n/n_\mathrm{p})_\mathrm{peak}$ and $|b|$ can only be given by a description that takes into account the dynamics of all electrons of the bunch and the pump depletion. However, since these two numbers have values between zero and one, the scaling laws of the acceleration gradient G may well be taken as those of the factor g that, in turn, depends only on the plasma density n_p and on the wavelength λ_w and magnitude B_w of the wiggler field.

The parameter g is represented in Figure 3 as a function of the plasma density n_p for a fixed value of the

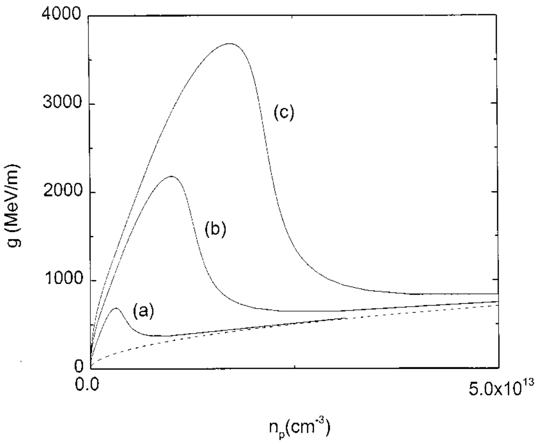

Fig. 3. The factor g in MeV/m as a function of the plasma density n_p in cm^{-3}, with $\lambda_\mathrm{w} = 5$ cm and $\omega/ck = 1$. The solid curves (a, b and c) are for $B_\mathrm{w} = 2, 6$ and 10 T, while the dashed line gives g in the non magnetic case, *i.e.*, $g = 10^{-4}(\omega/ck)\sqrt{n_\mathrm{p}(\mathrm{cm}^{-3})}$ MeV/m.

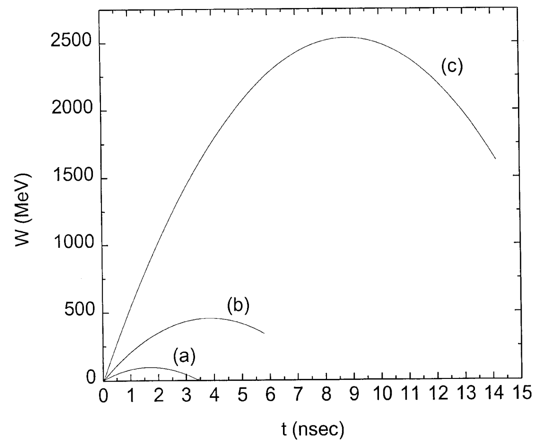

Fig. 4. Beam average energy w *vs.* time for infinitely long beam and the following values: $n_\mathrm{p} = 10^{14}$ cm^{-3}, $\omega/ck = 1$, $\lambda_\mathrm{w} = 2$ cm, $n_\mathrm{b} = 10^{11}$ cm^{-3}, $(\delta n/n_\mathrm{p})_\mathrm{peak} = 0.6$, $\tilde{N} = 200$ and $\gamma_j(0) = \gamma_0 = 10$. Curve (a) gives the non magnetic case in which $a_{\mathrm{w}0} = 0$ ($B_\mathrm{w} = 0$), $\omega/\omega_\mathrm{p} = 1$, $\lambda = 0.33$ cm; curve (b) is for $a_{\mathrm{w}0} = 30$ ($B_\mathrm{w} = 15$ T), $\beta_{\mathrm{p}0} = 0.8$, $\omega/\omega_\mathrm{p} = 0.46$, $\lambda = 1.7$ cm; curve (c) for $a_{\mathrm{w}0} = 38(B_\mathrm{w} = 19$ T), $\beta_{\mathrm{p}0} = 0.96$, $\omega/\omega_\mathrm{p} = 0.19$, $\lambda = 1.8$ cm.

wiggler wavelength and for increasing values of B_w. This figure reveals one of the basic characteristics of the present acceleration scheme. It shows that by adding a strong periodic magnetic field one can accelerate a bunch of electrons with very high gradients at plasma densities much lower than those used in the existing plasma-based acceleration schemes. Also, since ω_p is smaller and the frequency of the wave smaller than ω_p, one can work with wavelengths much longer than those used in the experiments made so far.

The second factor $(\delta n/n_\mathrm{p})_\mathrm{peak}$ appearing in equation (37) has its maximum value at $t = 0$, since the wave amplitude of the pump can only decrease during the process. The last factor $|b|$ can be increased by operating with electron bunches whose lengths are smaller than the wavelength λ of the wave and it is to demonstrate this point that we need the complete space-time description of the process.

3 Numerical results and comments

3.1 Infinitely long electron beams

We first consider equations (32–34) when the electron bunch and the pulse of accelerating wave are so long that we may go to the limit of infinitely long bunches and wave pulses. We also assume that the amplitude of the relative density perturbation depends only on time, *i.e.*, that $M_\mathrm{L} = M_\mathrm{L}(t)$.

Since the equations are periodic with period 2π in the variables $\theta_j(t)$, an initial distribution of the variables $\theta_j(0)$ and $p_j(0)$ that is periodic with period 2π will stay periodic at all times. In particular, the number $\tilde{N}$ of electrons whose phases occupy, at $t = 0$, any length equal to 2π on the θ-axis will be the same at all times. Therefore, if we take the average of equation (32) over a length of the z-axis equal to the wavelength $\lambda = 2\pi/k$ of the accelerating wave the right hand side of the averaged equation will consistently depend only on the time variable t.

If we change, for convenience, from the instantaneous positions $z_j(t)$ of the electrons to their phases $\theta_j(t)$, equation (32) can be rewritten under the following simpler form

$$\frac{\mathrm{d}}{\mathrm{d}t}M_\mathrm{L}(t) = -\,\mathrm{i}S_1\left\{\frac{1}{\tilde{N}}\sum_{j=1}^{\tilde{N}}\mathrm{e}^{-\mathrm{i}\theta_j(t)}\right\} - \mathrm{i}S_2\left\{\frac{1}{\tilde{N}}\sum_{j=1}^{\tilde{N}}\frac{1}{\gamma_j(t)}\mathrm{e}^{-\mathrm{i}\theta_j(t)}\right\} \qquad (32')$$

where $S_{1,2} = (\tilde{N}/\lambda)C_{1,2}$ and $j = 1,\ 2, ...,\ \tilde{N}$.

At this point, we solve the initial value problem associated to equations (32', 33, 34) in which, at $t = 0$, $M_\mathrm{L} = M_\mathrm{L}(0)$ and the electron beam is "cold" (*i.e.*, $p_j(0) = p_0$) and unbunched ($b(0) = 0$). The results of the numerical integration of this problem have been presented in reference [18]. We give, in Figure 4, the acceleration results of Figure 2 of [18], here only recast in physical units. The figure shows the time behaviour of the average energy $W = mc^2\langle\gamma\rangle = mc^2\frac{1}{\tilde{N}}\sum_{j=1}^{\tilde{N}}\gamma_j(t)$ of the sample of $\tilde{N}$ electrons in a rather typical situation and for different, increasing values of the wiggler parameter $a_{\mathrm{w}0}$. We can see that when this factor is increased, *i.e.*, essentially, when we increase the strength of the magnetic field of the wiggler, the initial rate of acceleration and the maximum value of the energy gained by the beam at saturation both increase considerably.

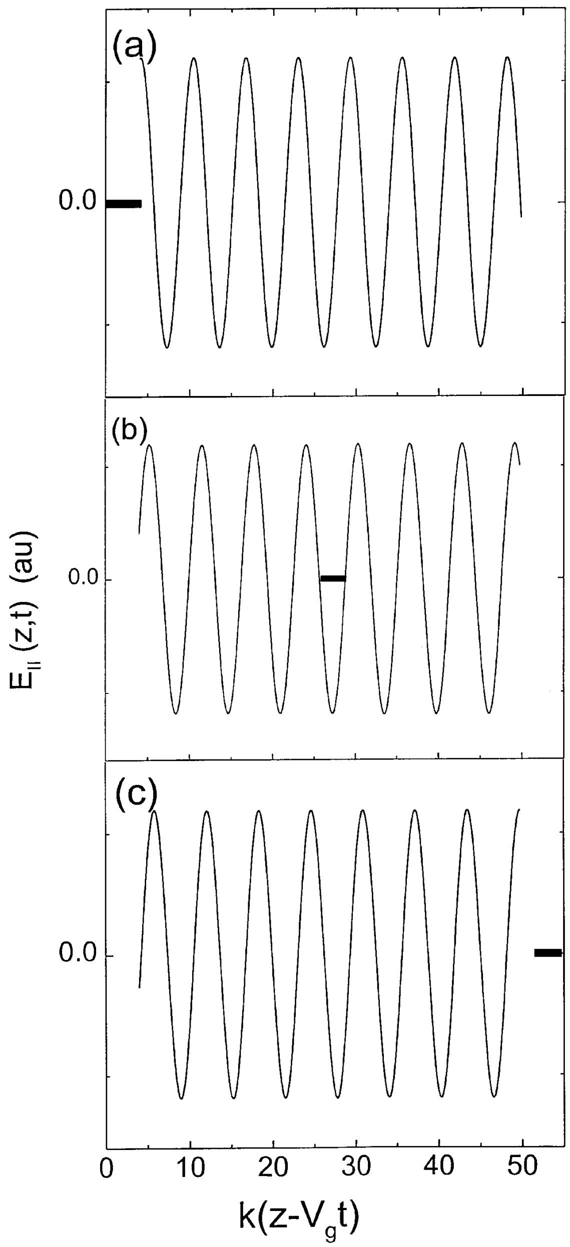

Fig. 5. Snapshots of the electron bunch crossing the accelerating region and in a reference frame which is moving with the group velocity $V_g = \omega'(k)$. The bunch comes from the left and is crossing the interaction length L_A with the pump wave. The curve (a) is at $t = 0$ and shows the bunch at the left of the wave packet just before the interaction. The other parts (b and c) show the electron bunch well inside the interaction region and when it has just been expelled at the right of the wave pulse.

3.2 Electron bunches of finite lengths

We solve the basic equations (32–34) and represent in Figure 5a the typical situation at $t = 0$. Both plasma and wiggler field are considered to extend over the whole z-axis, while the wave amplitude $M_L(z, t = 0)$ has the form of a step-function of length L_A. The figure shows the carrier

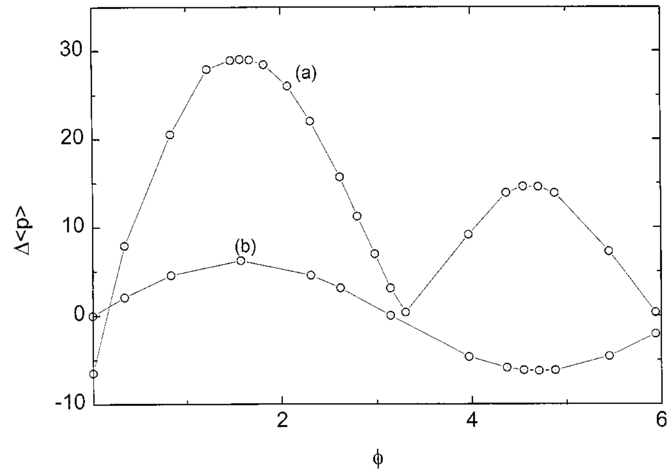

Fig. 6. Plot of the variation $\Delta\langle p\rangle$ in the average beam scaled momentum $\langle p\rangle$ *vs.* the phase Φ of the wave electric field at the injection point, $\Delta\langle p\rangle$ being defined as the value of $\langle p\rangle$ at the time $t \approx 2\pi/ck$ minus the value at $t = 0$, $p_0 = 50$. The phase Φ is that of the longitudinal field $E_{\|}(z,t) = -E_{\|0}\sin(kz-\omega t+\Phi)$, with $\omega/ck = 1$ and $n_b/n_p = 10^{-3}$. The figure has been made using an extremely short electron bunch, namely in the limit $kL_b \approx 0$. The curve (b) refers to the non magnetic case, in which $B_w = 0$ and $\omega = \omega_p$. Negative values of $\Delta\langle p\rangle$ mean that the bunch is decelerated. Curve (a) refers to a magnetised plasma with $\beta_{p0} = 0.996$, $\omega_p/ck_w = 4$, $k/k_w = 0.586$, and $(\delta n/n_p)_{peak} = 0.6$. The values given by curve (a) should be multiplied by ten.

wave at $t = 0$ inside the "acceleration length" L_A, with the electron bunch placed at the left of the wave pulse and outside it. The other parts of the figure show the bunch of electrons as it enters the wave packet (which is moving as a whole with its group velocity which has a small negative value in this case), is accelerated and finally expelled at the right side of the wave pulse.

The time history of the bunching factor of the short beam which is crossing the acceleration region L_A depends on a large number of parameters. First of all, there will be a strong dependence on the ratio L_b/λ between the bunch length L_b and the wavelength of the accelerating wave. For very long bunches, when $L_b/\lambda \gg 1$, we find that the bunching factor of the electrons inside the bucket increases abruptly from zero at the injection time up to values that, in all cases, do not exceed the value 0.7.

To reach larger values of the bunching factor, we must use electron bunches whose lengths are considerably smaller than the wavelength λ. We shall focus our description, here, only on these cases. For this kind of beams there is obviously a strong dependence on the relative phase between the wave and the bunch at the time of injection.

In Figure 6 we show how the energy ($\Delta\langle p\rangle$) gained by a short electron bunch over a distance within the acceleration length roughly equal to the wavelength, depends on the phase Φ of the electric field of the wave at the injection point. In the same figure we also give, by comparison, the values of the energy gained in the non magnetic case. One can see that the wiggler field not only allows the gain

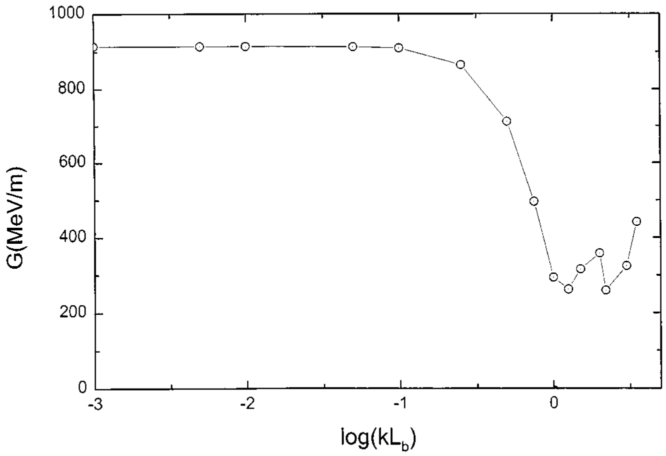

Fig. 7. Acceleration rate G as a function of $\log(kL_\mathrm{b}) = \log[2\pi(L_\mathrm{b}/\lambda)]$, with $\beta_{\mathrm{p}0} = 0.996, \omega_\mathrm{p}/ck_\mathrm{w} = 4, (\delta n/n_\mathrm{p})_\mathrm{peak} = 0.6$ and $k/k_\mathrm{w} = 0.586$, the injection value of $\langle p \rangle = p_0 = 50, n_\mathrm{b}/n_\mathrm{p} = 10^{-3}$ and $\omega/ck = 0.998$. The value of Φ at the injection point is $\pi/2$.

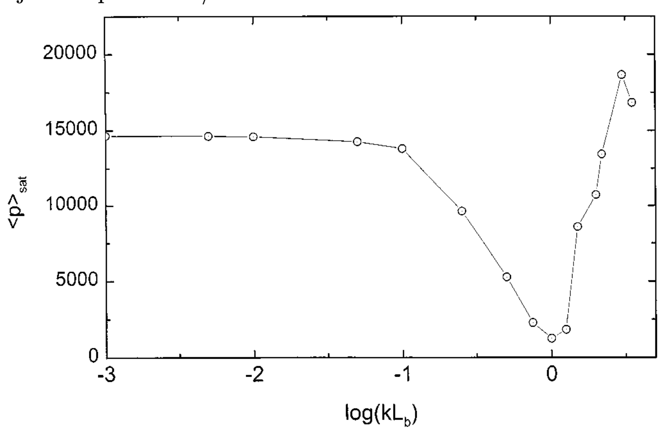

Fig. 8. Saturation value of the average momentum $\langle p \rangle$ as a function of kL_b. Same parameters as in Figure 7.

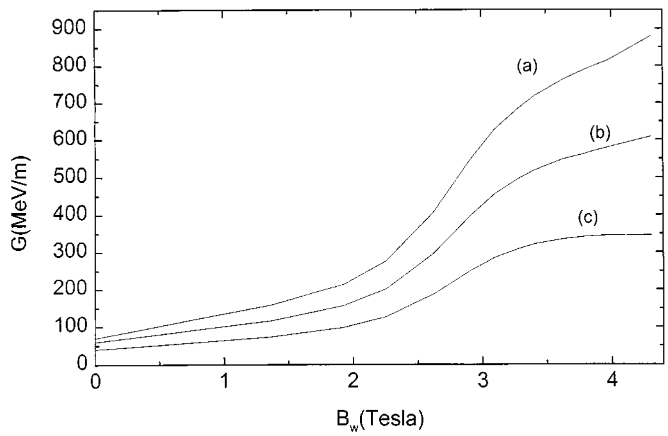

Fig. 9. Acceleration rate G *vs.* the applied wiggler field B_w in Tesla. Here $\omega/ck = 0.998$, $kL_\mathrm{b} = 0.05$ and the spatial period of the wiggler is $\lambda_\mathrm{w} = 6.28$ cm. Other parameters as in Figure 7. Curve (a) is for the injection phase $\Phi = \pi/2$, curve (b) for $\Phi = \pi/4$ and (c) for $\Phi = 0$.

of considerably more energy during the very first stage of the process but that it seems to be surprisingly able to accelerate also those bunches that have been injected with the "wrong" phases. This fact is presumably due to the transverse wiggling in the electron motion which is caused by the wiggler field.

The following two Figures 7 and 8 show how the increase in the beam average energy depends on the ratio L_b/λ, the bunches being injected in such a way that the leading electrons of the bunch see the "optimum" phase $\Phi = \pi/2$. Figure 7 gives the actual rate of acceleration G over a distance roughly equal to the wavelength as in the previous Figure 6, *versus* kL_b and shows that the gradient G saturates at a definite value for progressively smaller values of L_b/λ. It then decreases smoothly as the length of the bunch becomes comparable with the wavelength of the wave, when L_b/λ is between 0.01 and 0.1, roughly, in the case of the figure. When the length of the bunch is of the order of $\lambda/4$, the acceleration rate changes to a more complex behaviour which is probably due to the fact that the bunch is already too long and the trailing electrons begin to feel decelerating effects when they are injected into the wave pulse. This causes the breaking of the bunch into smaller bunches that move to the nearby buckets at the left of the leading electrons, and a final considerable spread in phase space.

Figure 8 gives the value of the average beam scaled momentum $\langle p \rangle = \frac{1}{N}\sum_{j=1}^{N} \beta_j(t)\gamma_j(t)$ at the time in which the energy gained by the beam is saturating and the beam itself starts feeding energy back into the wave. This energy also reaches smoothly a maximum value in the limit in which the ratio L_b/λ approaches zero. When the bunch length L_b is comparable with $\lambda/4$ one may get even higher values of the maximum energy gained by the beam, but, as we have just said, at the cost of a considerable spread of the bunch. It is, however possible, in these cases , to think of a process which would separate out from the whole beam a smaller bunch with the highest energy and possibly also with a good quality.

The last Figure 9 give the acceleration gradient G as a function of the strength of the wiggler field B_w, for different values of the phase of the carrier at the injection point.

4 Conclusion

In conclusion we have shown that the addition of a periodic magnetic field perpendicular to the beam motion in a plasma-based accelerator may lead to the following important consequences:

(i) The dispersion properties of the plasma change as a result of the strong diamagnetic currents that are induced by the external magnetic field and the consequent relativistic motion of the electrons of the plasma. According to equations (1, 36), this fact leads to the increase of the longitudinal part $E_\parallel$ of the electric field of the waves as well as of the rate of acceleration G by the factor ω_p/ω.

(ii) As shown in Figure 3, ultra-high gradient acceleration of electron beams can be obtained in low density plasmas. In addition, the accelerating waves have much longer wavelengths, even several cm long and, as a consequence, the ratio L_b/λ may have very small values, of the order of 10^{-2}, typically. The spread of the electron bunch in phase space during the strong acceleration may therefore be maintained at reasonably low levels.

(iii) Finally, the transverse wiggling motion of the electrons of the bunch that is induced by the external periodic field may lead to a considerable increase of the dephasing length between the bunch and the accelerating carrier wave and to much higher values of the energy gained by the beam at saturation. Also, as one can see from Figure 6, the wiggling motion seems to be able to influence positively the processes that take place at the injection point when the electron bunch enters into the accelerating pulse.

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